e-Edge Education Centre

Straight Lines

Class XI-EMM-MT-
LESSON-12

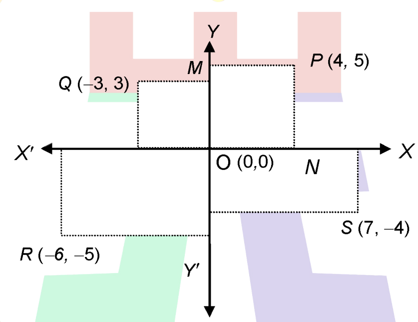
STRAIGHT LINES

1. PRELIMINARIES

The system of Geometry in which a point is specified by means of an ordered number-pair is known as Coordinate Geometry. It enables us to solve geometrical problems by algebraic methods.

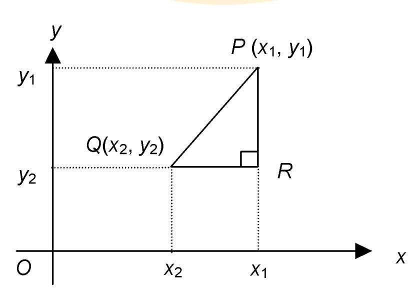
1. COORDINATES OF POINT

Let P be a point in a plane $X'OX, Y'OY$ are known as the coordinate axes. MP is the x-coordinate, called the abscissa, and NP is the y-coordinate called the ordinate of P. These two distances MP and NP together fix the position of P in the plane.



2. THE DISTANCE BETWEEN TWO GIVEN POINTS IN THE CARTESIAN SYSTEM

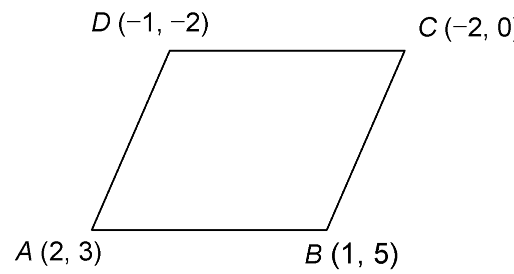
$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



Example: **Show that the points (2, 3) (1, 5) (−2, 0) and (−1, −2) are vertices of a parallelogram.**

Solution: Let the points be denoted by

A, B, C, D in order



$AB^2 = (2-1)^2 + (3-5)^2$

$= 1 + 4 = 5$

$BC^2 = (1+2)^2 + (5-0)^2$

$= 9 + 25 = 34$

$CD^2 = (-2+1)^2 + (0+2)^2$

$= 1 + 4 = 5$

$DA^2 = (-1-2)^2 + (-2-3)^2$

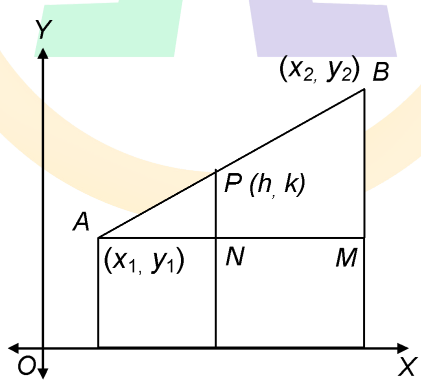
$= 9 + 25 = 34$

Since the opposite sides are equal, the figure drawn is a parallelogram.

2. SECTION FORMULA

Let A, B are the points (x_1, y_1), (x_2, y_2), here we show the coordinates of the point P on AB such that $AP : PB = l : m$.

Let the coordinates of P be (h, k)



$$\therefore\ k = \frac{ly_2 + my_1}{l+m}$$

$$h = \frac{lx_2 + mx_1}{l+m} \quad \text{(internally)}$$

Note: Putting $l = m$, we find that the mid-point of the line joining the points $(x_1, y_1), (x_2, y_2)$,

i.e. $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

When P lies outside AB i.e., external to AB such that $AP : BP = l : m$.

We have h = $\frac{lx_2 - mx_1}{l - m}, k = \frac{ly_2 - my_1}{l - m}$

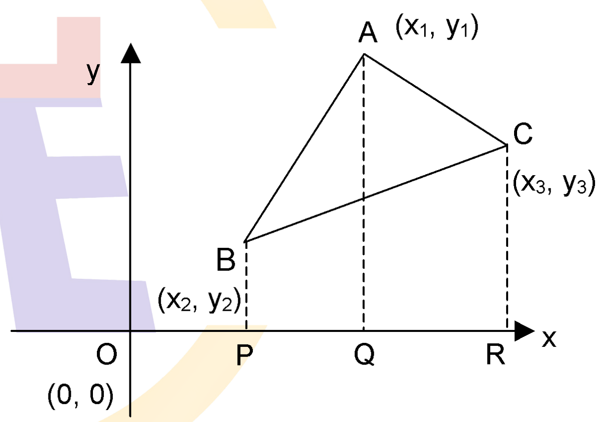
It has been assumed that l and m are positive numbers; if, however, we take $AP : BP$ to be positive in the former case and negative in the latter case, we have in both cases the formula

$$\boldsymbol{h = \frac{lx_2 + mx_1}{l + m}, k = \frac{ly_2 + my_1}{l + m}}$$

3. AREA OF A TRIANGLE

Let the vertices of the triangle be

$A(x_1, y_1),\ B(x_2, y_2),\ C(x_3, y_3)$



Drop perpendiculars form A, B and C on the x-axis.

Area of ΔABC = Area of trapezium ABPQ

+ Area of trapezium AQRC

− Area of trapezium BCRP

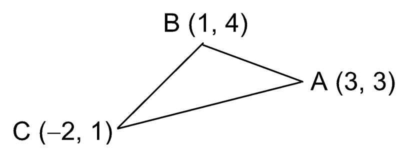
Area of $\Delta ABC = \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$

The area of a triangle is considered positive when its vertices are named in counter-clockwise order, and negative when named in clockwise order.

If three points are collinear, we have the area of the Δ = 0.

Example: **Find the area of the triangle with vertices (3, 3), (1, 4), (−2, 1).**

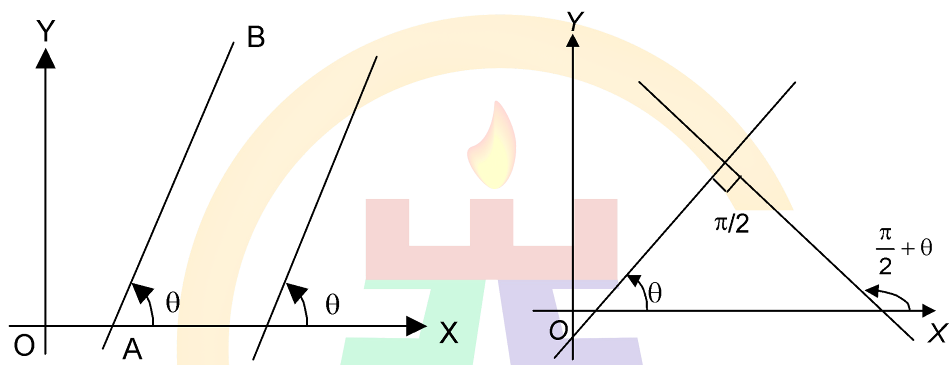
Solution: Let the given vertices be *A, B, C* respectively.



Area of $\Delta BAC = \frac{1}{2}[3(4-1)+1(1-3)-2(2-4)]$

$= \frac{1}{2}[9-2+2] = \frac{9}{2}$ sq. units

4. THE STRAIGHT LINE



SLOPE:

If a straight line *AB* makes an angle θ with the positive direction of the *x*-axis, tan θ is called the slope or gradient of the straight line and is usually denoted by the letter '*m*'.

It follows that

(i) if two lines are parallel, their slopes are equal, for the lines must be equally inclined to the positive direction of the *x*-axis.

(ii) if two lines are perpendicular the product of their slopes is −1, for if one line is inclined at an angle θ to the *x*-axis, the other must be inclined at $\frac{\pi}{2}+\theta$, hence their slopes are tan θ and tan $\left(\frac{\pi}{2}+\theta\right)$, i.e., tan θ and −cot θ.

∴ the product is −1.

Straight Lines

1. TO FIND THE SLOPE OF THE LINE JOINING ANY TWO POINTS (x_1, y_1) AND (x_2, y_2)

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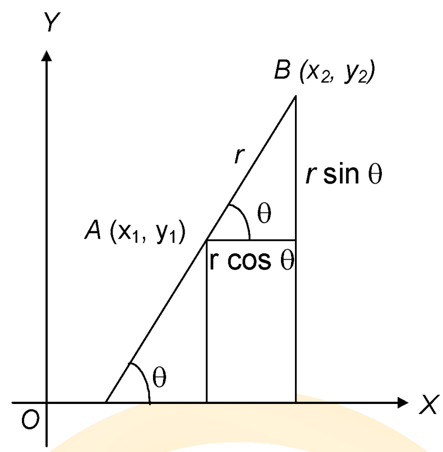
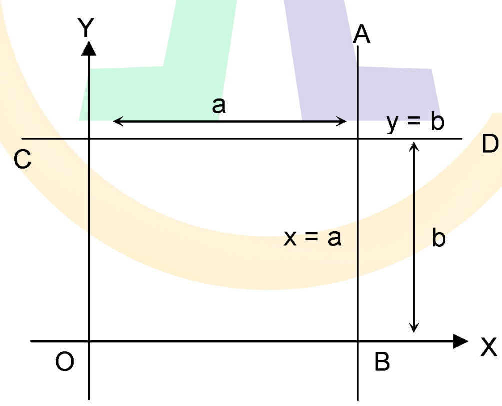


$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\text{difference of the ordinates of the two points}}{\text{difference of the abscissa}}$$

2. LINES PARALLEL TO THE CO ORDINATE AXES

Let the line *AB* be parallel to the *Y*-axis and at a distance '*a*' from it. Every point on *AB* will have its abscissa 'a', and hence the equation of *AB* is $x = a$.



Similarly, the equation of the straight line *CD* parallel to the *X*-axis and at a distance '*b*' from it is $y = b$.

3. ANGLE BETWEEN TWO GIVEN STRAIGHT LINES

Let *AB, AC* have slopes m_1, m_2 and be inclined to the *X*-axis at θ_1, θ_2.

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Then $\angle = \theta_1 - \theta_2 = \theta$

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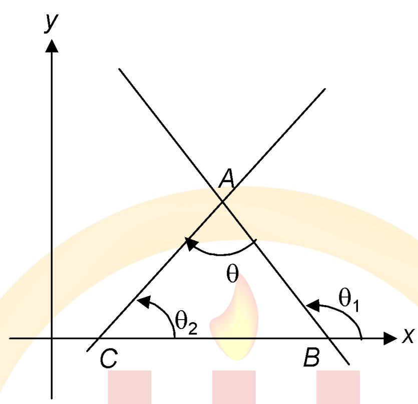
$\therefore \tan \angle CAB = \dfrac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$

i.e., **tan θ =** $\dfrac{m_1 - m_2}{1 + m_1 m_2}$

where θ is the angle between *AB* and *AC*



Note: It is customary to take θ, as the acute angle between the two lines, and hence mostly one can take the above formula as tan θ = $\left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

If the lines are parallel,

tan θ = 0, since θ = 0

$\therefore m_1 - m_2 = 0$

$\therefore \boldsymbol{m_1 = m_2}$

and **if the lines are perpendicular,** tan θ is not defined, since $\theta = \dfrac{\pi}{2}$, and

therefore $m_1 m_2 + 1 = 0$

$\therefore \boldsymbol{m_1 m_2 = -1}$.

Example: **Find the acute angle between the two lines with slopes $\dfrac{1}{5}$ and $\dfrac{3}{2}$.**

Solution: If the angle between the lines is θ,

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$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{1}{5} - \frac{3}{2}}{1 + \left(\frac{1}{5}\right) x \left(\frac{3}{2}\right)}\right| = |-1| = 1$$

Therefore the angle is 45°.

5. VARIOUS FORMS OF THE EQUATION OF A STRAIGHT LINE

To represent a straight line we need to be given two independent information; and depending upon the two, the equation also changes.

We have therefore the following forms of equations to straight lines:

1. SLOPE ONE POINT FORM

Given that *a* line has a slope ($m = \tan\theta$) which gives the direction it may be noted that '*m*' alone does not give the equation of the line and with the same slope there can be any number of straight lines all of which are parallel. Given that it passes through a given point (x_1, y_1). In this case the equation has the form.

$$y - y_1 = m(x - x_1)$$

Example: **If a line has a slope = $\frac{1}{2}$ and passes through *(− 1, 2)*; find its equation.**

Solution: The equation of the line is

$$y - 2 = \frac{1}{2}(x - (-1))$$

i.e., $2y - 4 = x + 1$

i.e., $x - 2y + 5 = 0$

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2. Y-INTERCEPT FORM

Given the slope '*m*' and the length '*c*' (called the intercept) cut off on the *y*-axis by the line. In this case the form of the equation is

$$\boldsymbol{y = mx + c}$$

Example: **If a line has a slope $\frac{1}{2}$ and cuts off along the positive *y*-axis of length $\frac{5}{2}$ find the equation of the line.**

Solution: $y = \frac{1}{2}x + \frac{5}{2}$

i.e., $2y = x + 5$

i.e., $x - 2y + 5 = 0$

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3. TWO POINT FORM

Equation of a line passing through two points (x_1, y_1) and (x_2, y_2).

In this case the equation to the line is of the form

$$\boldsymbol{\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}}$$

Where $x_1 \neq x_2$ and $\left(\frac{y_1 - y_2}{x_1 - x_2}\right)$ is the slope (m) of the line.

Example: **If a line passes through two points (1, 5) and (3, 7), find its equation.**

Solution: $\frac{y-5}{x-1} = \frac{-7+5}{-3+1} = \frac{-2}{-2} = 1$

i.e., $y - 5 = x - 1$

i.e., $x - y + 4 = 0$

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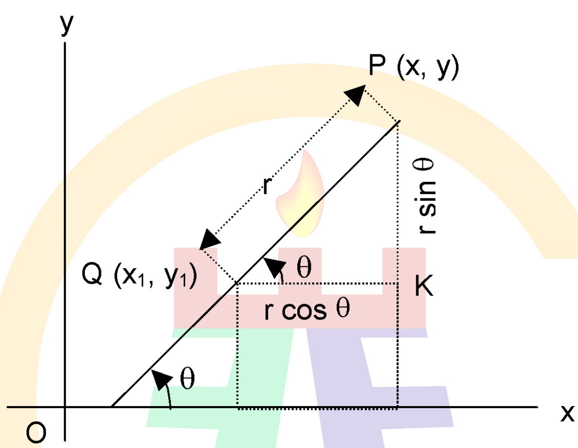
4. SYMMETRIC FORM

If (x_1, y_1) is a given point on a straight line, (x, y) any point on the line, θ the inclination of the line and r the distance between the points (x, y) and (x_1, y_1) then $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}$

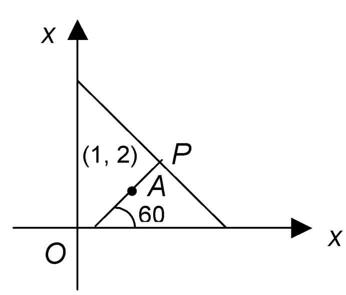


is called the symmetric form of the equation to a line.

Example: **A straight line passes through a point A (1, 2) and makes an angle 60° with the x-axis. This line intersects the line $x + y = 6$ at the point *P*. Find AP.**

Solution: Let required line be

$$\frac{x-1}{\cos 60^\circ} = \frac{y-2}{\sin 6^\circ} = r(\equiv AP)$$



∴ P is $(1 + r\cos 60^\circ, 2 + r\sin 60^\circ)$ which lies on $x + y = 6$

The condition is

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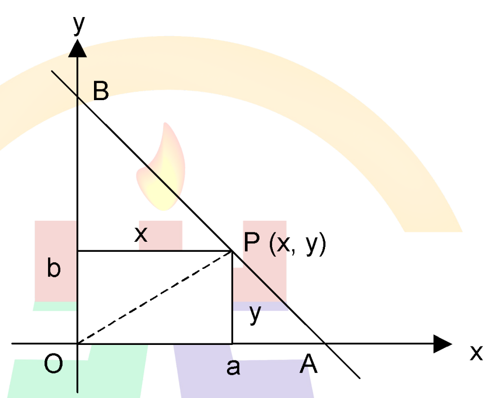
$$\left(1+\frac{r}{2}\right)+\left(2+r\frac{\sqrt{3}}{2}\right)=6 \text{ or } r\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)=3$$

$$\therefore r=\frac{6}{1+\sqrt{3}}=\frac{6\left(1-\sqrt{3}\right)}{\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)}=\frac{6\left(1-\sqrt{3}\right)}{\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)}=\frac{6\left(1-\sqrt{3}\right)}{-2}=3\left(\sqrt{3}-1\right).$$

5. INTERCEPT FORM

Let the line make intercepts *a* and *b* on the axes of *x* and *y* respectively.

Let *P*(*x, y)* represent any point on the line. Join *OP.*



We have,

Area of triangle *OAB* = area of triangle *OAP* + area of triangle *OPB*

i.e., $\frac{1}{2}ab=\frac{1}{2}ay+\frac{1}{2}bx$

or $\boldsymbol{\frac{x}{a}+\frac{y}{b}=1}$

This is called the intercept form of the equation of straight line.

Example: **Find the equation of the straight line, which passes through the point (3, 4) and whose intercept on *y*-axis is twice that on *x*-axis.**

Solution: Let the equation of the line be

$\frac{x}{a}+\frac{y}{b}=1$ …(i)

According to the question $b = 2a$

∴from (i) equation of line will become

$\frac{x}{a}+\frac{y}{2a}=1$ or $2x + y = 2a$ ….(ii)

Since line (ii) passes through the point (3, 4)

∴ $2 \times 3 + 4 = 2a$ ∴ $a = 5$

∴ from (ii), equation of required line will be

$2x + y = 10$.

6. NORMAL FORM

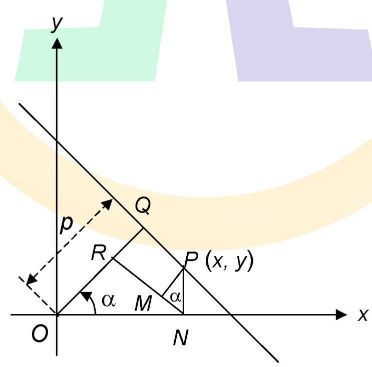
If p is the length of the perpendicular from the origin upon a straight line, and that α is the angle the perpendicular makes with the axis of x, equation of the straight line can be obtained as

∴ $x \cos\alpha + y \sin\alpha = p$

This is called the perpendicular form.

Suppose $\frac{x}{a}+\frac{y}{b}=1$ is the equation of the straight line

In this case, $a=\frac{p}{\cos\alpha}, b=\frac{p}{\sin\alpha}$



∴ by substitution, the equation becomes

$$\frac{x\cos\alpha}{p}+\frac{y\sin\alpha}{p}=1$$ or $\boldsymbol{x\cos\alpha + y\sin\alpha = p}$

This is known as the normal form of the equation to a straight line.

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Example: **Find the equation of the straight line upon which the length of perpendicular from origin is $3\sqrt{2}$ units and this perpendicular makes an angle of 75° with the positive direction of *x*-axis.**

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Solution: Let *AB* be the required line and *OL* be perpendicular to it.

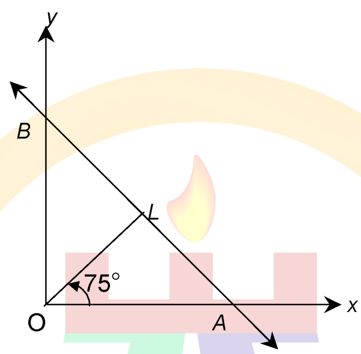
Given $OL = 3\sqrt{2}$ and $\angle LOA = 75°$

∴ equation of line *AB* will be

$x \cos 75° + y \sin 75° = 3\sqrt{2}$

[Normal form] ...(i)

Now $\cos 75° = \cos(30° + 45°) = \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} - \frac{1}{2}.\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$



and $\sin 75° = \sin(30° + 45°) = \sin 30° \cos 45° + \cos 30° \sin 45° = \frac{1}{2}.\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

∴ from (i) equation of line *AB* is

$x\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + y\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) = 3\sqrt{2}$ or $(\sqrt{3}-1)x + (\sqrt{3}+1)y - 12 = 0$

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7. GENERAL FORM

Any linear (1st degree in *x* and *y*) equation of the form

$Ax + By + C = 0$

represents a straight line.

The straight line in this form, has

(a) slope $= m = -\frac{A}{B} = -\frac{\text{coefficient of } x}{\text{coefficient of } y}$

(b) x intercept = $-\frac{C}{A}$ and

y intercept = $-\frac{C}{B}$

Example: **Find the value of *k* so that the straight line $2x + 3y + 4 + k(6x - y + 12) = 0$ and $7x + 5y - 4 = 0$ are perpendicular to each other.**

Solution: Given lines are

$(2 + 6k)\, x + (3 - k)\, y + 4 + 12k = 0$ …(i)

and $7x + 5y - 4 = 0$ …(ii)

slope of line (i), $m_1 = -\frac{2+6k}{3-k} = \frac{2+6k}{k-3}$

and slope of line (ii), $m_2 = -\frac{7}{5}$

Since line (i) is perpendicular to line (ii)

$\therefore \left(\frac{2+6k}{k-3}\right)\left(-\frac{7}{5}\right) = -1$

or $(2 + 6k)\, 7 = 5\,(k - 3)$

or $14 + 42k = 5k - 15$ or $37k = -29$

or $k = -\frac{29}{37}$

6. CONDITION OF COLLINEARITY

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Three points *A, B* and *C* or more are said to be collinear when they lie on the same line.

Collinearity can be checked in two simple ways.

(i) If Slope of *AB* = slope of *BC*, points *A, B, C* are collinear.

(ii) If Area of Δ*ABC* = 0, points *A, B, C* are collinear.

(iii) Collinear points must satisfy section formulae.

Example: **Prove that the points (1, 2), (-3, λ) and (4, 0) are collinear if λ is equal to $\frac{14}{3}$.**

Solution: Let *A* (1,2), *B* (-3, λ) and *C* (4, 0) to be the three points.

Slope of *AB* = slope of *AC*

$$\Rightarrow \frac{\lambda - 2}{-3-1} = \frac{2-0}{1-4} \Rightarrow \frac{\lambda - 2}{-4} = -\frac{2}{3} \Rightarrow \lambda = 2 + \frac{8}{3} = \frac{14}{3}$$

Alternatively,

$$\text{Area } (\Delta\, ABC) = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ -3 & \lambda & 1 \\ 4 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 4(2-\lambda) + (\lambda + 6) = 0 \Rightarrow -4\lambda + \lambda + 8 + 6 = 0 \Rightarrow \lambda = \frac{14}{3}$$

7. DISTANCE OF A POINT FROM A LINE

The perpendicular distance of a point from a line can be obtained when the equation of the line and the coordinates of the point are given.

Case I :

Let us first derive the formula for this purpose when the equation of the line is given in normal form.

Let the equation of the line *l* in normal form be

$x \cos \alpha + y \sin \alpha = p$,

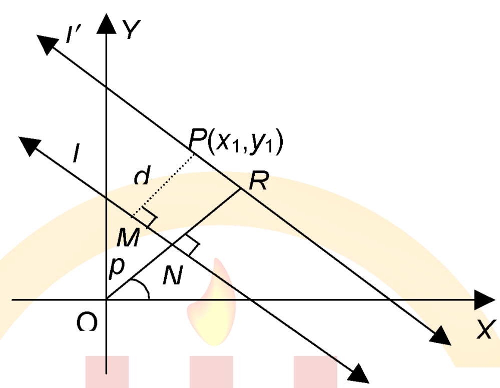
when α is the angle made by the perpendicular from origin to the line with positive direction of x-axis and *p* is the length of this perpendicular. Let $P(x_1, y_1)$ be the point, not on the line *l*. Let the perpendicular drawn from the point *P* to the line *l* be *PM* and *PM* = *d*. Point P is assumed to lie on opposite side of the line l from the origin O. Draw a line l' parallel to the line l through the point P. Let ON be perpendicular from the origin to the line l which meets the line l' in point R.

Obviously

ON = p and $\angle XON = \alpha$

Also, from the Figure, we note that

$OR = ON + NR = p + MP$



Therefore, length of the perpendicular from the origin to the line l' is

$OR = p + d$

and the angle made by the perpendicular OR with positive direction of x-axis is α. Hence, the equation of the line l' in normal form is

$x \cos \alpha + y \sin \alpha = p + d$

Since the line l' passes through the point P, the coordinates (x_1, y_1) of the point P should satisfy the equation of the line l' giving

$x_1 \cos \alpha + y_1 \sin \alpha = p + d$,

or $\quad d = x_1 \cos \alpha + y_1 \sin \alpha - p$.

The length of a segment is always non−negative. Therefore, we take the absolute value of the RHS, i.e.,

$d = | x_1 \cos \alpha + y_1 \sin \alpha - p |$.

Thus, the length of the perpendicular is the absolute value of the result obtained by substituting the coordinates of point P in the expression $x \cos \alpha + y \sin \alpha - p$

Case II:

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Let the equation of the line be

$Ax + By + C = 0$

Reducing the general equation to the normal form, we have

$$\pm\frac{A}{\sqrt{A^2+B^2}}x \pm \frac{B}{\sqrt{A^2+B^2}}y = \pm\frac{C}{\sqrt{A^2+B^2}}$$

Where sign is taken + or − so that the RHS is positive.

(a) When $C < 0$

In this case the normal form of the equation of the line / becomes

$$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y = -\frac{C}{\sqrt{A^2+B^2}}$$

Now, from the result of case (I), the length of the perpendicular segment drawn from the point $P(x_1, y_1)$ to the line (II) is

$$d = \left|\frac{A}{\sqrt{A^2+B^2}}x_1 + \frac{B}{\sqrt{A^2+B^2}}y_1 + \frac{C}{\sqrt{A^2+B^2}}\right|$$

i.e.,

$$d = \left|\frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}}\right|$$

(b) When $C > 0$

In this case the normal form of the equation (II) becomes

$$-\frac{A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}}$$

Again, by the result of case (I), the distance d is

$$d = \left|\frac{-Ax_1 - By_1 - C}{\sqrt{A^2+B^2}}\right|$$

or $$d = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right|$$

The perpendicular distance of (x_1, y_1) from the line $ax + by + c = 0$ is given by

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

So, Perpendicular distance from origin is $\frac{|c|}{\sqrt{a^2 + b^2}}$.

Example: **The equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is $(2, -1)$. Find the length of side of the triangle.**

Solution: Equation of side BC is

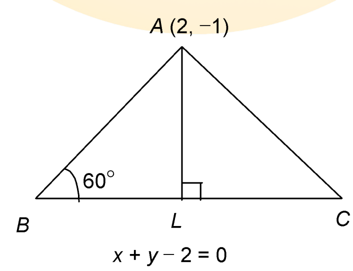
$x + y - 2 = 0$ ...(i)

$A \equiv (2, -1)$

Now $AL = \left| \frac{2 - 1 - 2}{\sqrt{1^2 + 1^2}} \right| = \frac{1}{\sqrt{2}}$

From ΔABL, $\sin 60° = \frac{AL}{AB}$

$\therefore \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}AB}$ or $AB = \sqrt{\frac{2}{3}}$



8. DISTANCE BETWEEN PARALLEL LINES

The distance between $ax + by + c = 0$ and $ax + by + c' = 0$

is given by, $d = \dfrac{|c - c'|}{\sqrt{a^2 + b^2}}$

Make sure that coefficients of *x* and *y* are same in both the lines before applying the formula.

Example: **Find the distance between the lines 5*x* + 12*y* + 40 = 0 and 10*x* + 24*y* – 25 = 0.**

***Solution*:** Here, coefficients of *x* and *y* are not the same in both the equations. So, we write them as

$5x + 12y + 40 = 0$

$5x + 12y - \dfrac{25}{2} = 0$

Now, distance between them = $\dfrac{\left|40 - \left(-\dfrac{25}{2}\right)\right|}{\sqrt{(5)^2 + (12)^2}} = \dfrac{105}{2(13)} = \dfrac{105}{26}$