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LESSON

RELATIONS AND FUNCTIONS

1. RELATIONS

DEFINITION

A relation *R*, from a non-empty set *A* to another non-empty set *B*, is a subset of $A \times B$

Any subset of $A \times B$ is relation from *A* to *B*.

R is a relation from *A* to *B* $\Leftrightarrow$ $R \subseteq A \times B$

$\Leftrightarrow$ $R \subseteq \{(a, b) : a \in A,\ b \in B\}$

Example: Let $A = \{1, 2\}, B = \{a, b, c\}$

Let $R = \{(1,\ a), (1,\ c)\}$

Here *R* is a subset of $A \times B$ and hence it is a relation from *A* to *B*.

2. DOMAIN AND RANGE OF A RELATION

1. DOMAIN OF A RELATION

Let *R* be a relation from *A* to *B*. The domain of relation *R* is the set of all those elements $a \in A$ such that $(a, b) \in R$ for some $b \in R$.

2. RANGE OF A RELATION

Let R be a relation from A to B. The range of R is the set of all those elements $b \in R$ such that $(a, b) \in R$ for some $a \in A$.

Set B is called as codomain of relation R.

Example: Let $A = \{2, 3, 5\}$ and $B = \{4, 7, 10, 8\}$

Let $aRb \Leftrightarrow a$ divides b

Then $R = (2, 5)$ and range of $R = \{4, 10, 8\}$

Codomain of $R = B = \{4, 7, 10, 8\}$

3. REPRESENTATION OF A RELATION

1. ROSTER FORM

In this form a relation R is represented by the set of all ordered pairs belonging to R.

Example: Let $A = \{-1, 1, 2\}$ and $B = \{1, 4, 9, 10\}$

Let aRb means $a^2 = b$

Then R (in roaster form) = $\{(-1, 1), (1, 1), (2, 4)\}$

2. SET-BUILDER FORM

In this form, the relation R is represented as $\{(a, b) : a \in A, b \in B, a.....b\}$, the blank is to be replaced by the rule which associates a to b.

Example: Let $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8\}$

Let $R = \{(1, 2), (3, 4), (5, 6), (7, 8)\}$, then R in the builder form can be written as

$$R = \{(a, b) : a \in A,\ b \in B;\ a - b = -1\}$$

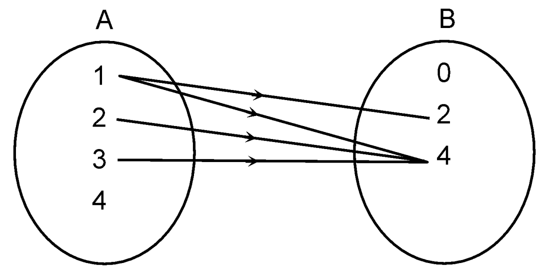
3. BY ARROW DIAGRAM

In this form, the relation R is represented by drawing arrows from first component to the second component of all ordered pairs belonging to R.

Example: Let $A = \{1, 2, 3, 4\}$, $B = \{0, 2, 4\}$ and R be relation 'is less than' from A to B, then

$$R = \{(1, 2), (1, 4), (2, 4), (3, 4)\}$$

This relation R from A to B can be represented by the arrow diagram as shown in the figure.



4. TOTAL NUMBER OF RELATIONS

Let A and B be two non empty finite sets having p and q elements respectively.

Then $n(A \times B) = n(A).n(B) = pq$

Therefore, total number of subsets of $A \times B = 2^{pq}$

Since each subset of $A \times B$ is a relation from A and B, therefore total number of relations form A to B is 2^{pq}

Example: Let $A = \{1, 2\}$, $B = \{3, 4, 5\}$

Then $n(A \times B) = n(A).n(B) = 2 \times 3 = 6$

$\therefore$ Number of relations from A to B = 2^6 = 64

Example: **If R is the relation 'is less than' from $A = \{1, 2, 3, 4, 5\}$ to $B = \{1, 4, 5\}$, write down the Cartesian product corresponding to R. Also find R^{-1} (aRb is a relation then bR' a is relation inverse to R i.e. $R' = R^{-1}$).**

Solution: Clearly, $R = \{(a, b) \in A \times B : a < b\}$

$\therefore$ $R = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

So, $R^{-1} = \{(4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4)\}$.

Example: **Let $A = \{3, 5\}$, $B = \{7, 11\}$**

Let $R = \{(a, b) : a \in A, b \in B, a - b$ is even$\}$

Show that R is an universal relation from A to B.

Solution: Given, $= \{3, 5\}, B = \{7, 11\}$

Now, $= \{(a, b) : a \in A, b \in B$ and $a - b$ is even$\}$ = (3, 7), (3, 11), (5, 7), (5, 11)}

Also $A \times B = \{(3, 7), (3, 11), (5, 7), (5, 11)\}$

Clearly, $R = A \times B$

Hence R is an universal relation from A to B.

5. DEFINITION OF A FUNCTION

- **Definition 1**

A function *f* is a relation from a non-empty set *A* to a non-empty set *B* such that domain of *f* is *A* and no two distinct ordered pairs in *f* have the same first element.

- **Definition 2**

Let *A* and *B* be two non-empty sets, then a rule of which associates each element of *A* with a unique element of *B* is called a mapping or a function from *A* to *B* we write $f : A \to B$ (read as *f* is a function from *A* to *B*).

For a function from *A* to *B*:

(i) *A* and *B* should be non-empty.

(ii) Each element of *A* should have image in *B.*

(iii) No element of *A* should have more than one images in *B*.

Example: **Examine each of the following relations given below and state in each case, giving reasons whether it is a function or not?**

(i) $R = \{(1, 2), (2, 2), (3, 1), (4, 2)\}$

(ii) $R = \{(2, 2), (1, 2), (1, 4), (4, 4)$

(iii) $R = \{(1, 2), (2, 3), (4, 5), (5, 6), (6, 7)\}$

Solution: (i) Since first element of each ordered pair is different, therefore this relation is a function.

(ii) Since the same first element 1 corresponds to two different images 2 and 4, hence this relation is not a function.

(iii) Since first element of each ordered pair is different, therefore this relation is a function.

6. DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION

The set *A* is called as the domain of the map *f* and the set *B* is called as the co-domain. The set of the images of all the elements of *A* under the map *f* is called the range of *f* and is denoted by $f(A)$.

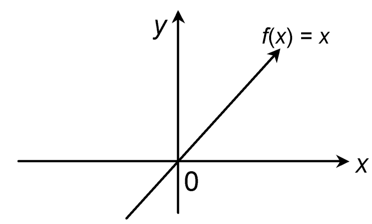
Thus range of *f* i.e. $f(A) = \{f(x) : x \in A\}$.

Clearly $f(A) \subseteq B$

7. IMPORTANT FUNCTIONS AND THEIR GRAPHS

Polynomial functions: $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx^n$, where $a_0, a_1, a_2, \ldots\ldots, a_n \in \boldsymbol{R}$ is said to be *a* polynomial function of degree *n*.

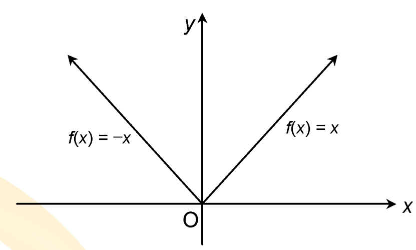
Identity function: An identity function in *x* is defined as $f : \boldsymbol{R} \to \boldsymbol{R}$, $f(x) = x$.



Absolute value function: An absolute value function in *x* is defined as $f : \boldsymbol{R} \to \boldsymbol{R}$, $f(x) = |x|$.

$$y = f(x) = |\,x\,| = \begin{cases} -x\,, & x \le 0 \\ x\,, & x > 0 \end{cases}$$

Domain : ***R*** ; Range : $[0, \infty)$;



Greatest integer function (step function): The function $f(x) = [x]$ is called the greatest integer function and is defined as follows:

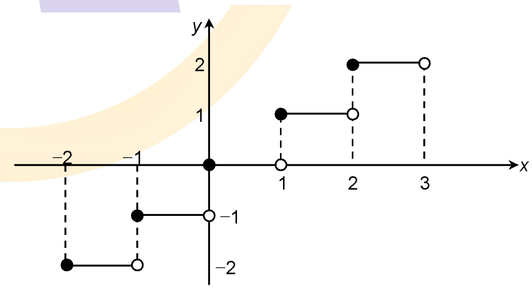
$[x]$ is the greatest integer less than or equal to x .

Then $[x] = x$ if x is an integer

= integer just less than x if x is not an integer.

$y = f(x) = [\,x\,]$

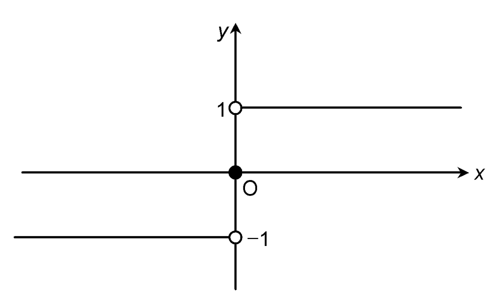
Domain : ***R*** ; Range : ***I***



Signum function: The function is defined as

$y = f(x) = \text{sgn}\,(x)$

$$\text{sgn}\,(x) = \begin{cases} \dfrac{|\,x\,|}{x}\,, & x \ne 0 \\ 0\,, & x = 0 \end{cases}$$



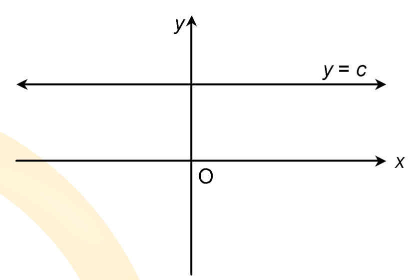
or $\text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$

Domain : $\boldsymbol{R}$; Range → {−1, 0, 1}

Rational algebraic function: A function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function.

The domain of a rational function $\frac{p(x)}{q(x)}$ is the set of all real numbers except points where $q(x) = 0$.

Constant function: The function defined as $f : \boldsymbol{R} \to \{c\}$ where $f(x) = c$



8. ALGEBRAIC OPERATIONS ON FUNCTIONS

Let us consider two functions.

$f : D_1 \to \boldsymbol{R}$ and $g : D_2 \to \boldsymbol{R}$. We describe functions $f + g$, $f - g$, $f.g$ and f/g as follows:

- $f + g : D \to \boldsymbol{R}$ is a function defined by
  $(f + g)\, x = f(x) + g(x)$, where $D = D_1 \cap D_2$
- $f - g : D \to \boldsymbol{R}$ is a function defined by
  $(f - g)\, x = f(x) - g(x)$, where $D = D_1 \cap D_2$
- $f\,.\,g : D \to \boldsymbol{R}$ is a function defined by
  $(f\,.\,g)\, x = f(x)\,.\,g(x)$, where $D = D_1 \cap D_2$
- $f\,/\,g : D \to \boldsymbol{R}$ is a function defined by
  $(f\,/\,g)\, x = \frac{f(x)}{g(x)}$, where $D = D_1 \cap \{x \in D_2 : g(x) \neq 0\}$
- $(\alpha f)(x) = \alpha f(x),\ x \in D_1$ and α is any real number.

Example: **If $f : R \to R$ is defined by $f(x) = x^3 + 1$ and $g : R \to R$ is defined by $g(x) = x + 1$, then find $f + g,\ f - g,\ f\,.g,\ \frac{f}{g}$ and $\alpha f\ (\alpha \in R)$.**

Solution: $f+g:R\to R$ is defined by $(f+g)(x)=f(x)+g(x)=x^3+1+x+1=x^3+x+2$

$f-g:R\to R$ is defined by $(f-g)(x)=f(x)-g(x)=x^3+1-x-1=x^3-x$

$f.g:R\to R$ is defined by $(fg)(x)=f(x)g(x)=(x^3+1)(x+1)=x^4+x^3+x+1$

$\frac{f}{g}:R-\{-1\}\to R$ is defined by $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{x^3+1}{x+1}=\frac{(x+1)(x^2-x+1)}{x+1}=x^2-x+1$

$\alpha f:R\to R$ is defined by

$(\alpha f)(x)=\alpha f(x)=\alpha(x^3+1)=\alpha x^3+\alpha$

Example: **Let $f(x)=\sqrt{x}$ and $g(x)=x$ be two functions defined over the set of non-negative real numbers. Find $(f+g)(x), (f\to g)(x), (fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.**

Solution: Given $(f+g)(x)=\sqrt{x}+x$, $(f-g)(x)=\sqrt{x}-x$,

$(fg)\,x=\sqrt{x}(x)=x^{3/2}$ and $\left(\frac{f}{g}\right)(x)=\frac{\sqrt{x}}{x}=x^{-1/2},\ x\neq 0$