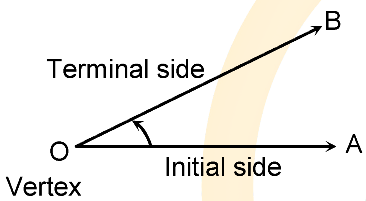
LESSON-3

TRIGONOMETRIC FUNCTIONS

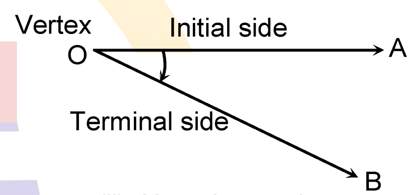
1. MEASUREMENT OF ANGLES

1. ANGLE

An angle is considered as figure obtained by rotating a given ray about its initial point. The original ray (before rotation) is known as **initial side** and the final position of the ray (after rotation) is known as **terminal side** of the angle, the point of rotation is known as **vertex**. The angle is said to be positive or negative as the rotation is anti-clockwise or clockwise.



(i) Positive angle



(ii) Negative angle

There are two methods used in measuring angles.

- **Sexagesimal System (Degree Measure)**

  The measure of an angle will be one degree, if the rotation from initial side to terminal side is $\left(\frac{1}{360}\right)^{th}$ of a revolution.

  1 right angle = 90 degrees (= 90°)

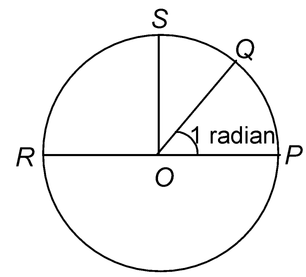
  1 degree = 60 minutes (= 60')

  1 minute = 60 seconds (= 60")

- **Circular System (Radian Measure)**

  One radian is the angle subtended at the centre of any circle by an arc of the circle equal in length to the radius.

2. RELATION BETWEEN DEGREES AND RADIANS

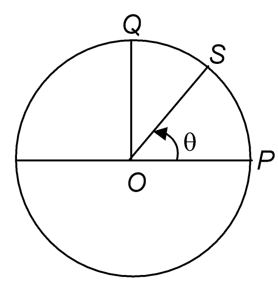


The angle subtended by a circle at its centre = $2\pi^c = 360°$

⇒ π radians = 180 degrees or $\pi^c = 180°$

⇒ $1^c = \left(\frac{180}{\pi}\right)^0 = 57°16'$ (Approx)

3. LENGTH OF AN ARC OF A CIRCLE



⇒ Length of arc = (radius of the circle) × (angle subtended by that arc at the centre of circle)

arc $PQ = r$, if the length of arc $PS = l$,

then $l = r\theta$

Example: **Find the radian measure corresponding to −37° 30′.**

Solution: ∵ $60' = 1°$ ⇒ $30' = \left(\frac{1}{2}\right)^\circ$ ⇒ $-37°\ 30' = -37\frac{1}{2}^\circ = -\frac{75}{2}^\circ$

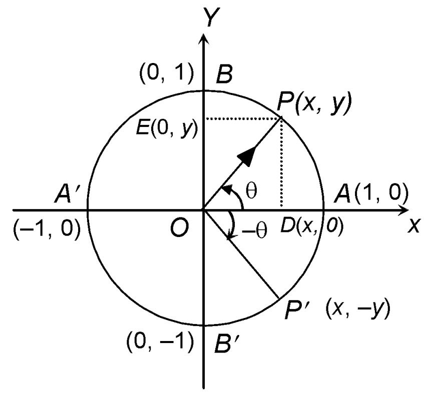
∵ $360° = 2\pi$ radians ⇒ $\frac{-75°}{2} = -\frac{2\pi}{360} \times \frac{75}{2}$ radians $= \frac{-5\pi}{24}$ radians

Example: **The minute hand of a clock is 10 cm long. How far does the tip of the hand move in 20 minutes ?**

Solution : The minute hand moves through 120° in 20 minutes or moves through $\frac{2\pi}{3}$ radians. Since the length of the minute hand is 10 cm, the distance moved by the tip of the hand is given by the formula $l = r\theta = 10 \times \frac{2\pi}{3} = \frac{20\pi}{3}$ cm.

2. TRIGONOMETRIC FUNCTIONS OF AN ANGLE

The six trigonometric ratios sine, cosine, tangent, cotangent, secant and cosecant of an angle θ, 0° < θ < 90° are defined as the ratios of two sides of a right-angled triangle with θ as one of the angles.



cosθ = x, the x coordinate of P

sinθ = y, the y coordinate of P

tanθ = $\frac{y}{x}, x \neq 0$

cotθ = $\frac{x}{y}, y \neq 0$

secθ = $\frac{1}{x},\ x \neq 0$

cosecθ = $\frac{1}{y},\ y \neq 0$

Angles measured anticlockwise from the initial line OX are deemed to be positive and angles measured clockwise are considered to be negative.

- **The range and sign of the trigonometric ratios in the four quadrants are depicted in the following table.**

In the second quadrant **(Only sine and cosecant are positive)**	**Y**	**In the first quadrant** **(All trigonometric ratios are positive)**
x′	**O**	**x**
In the third quadrant **(Only tangent and cotangent are positive)**	**Y′**	**In the fourth quadrant** **(Only cosine and secant are positive)**

1. TRIGONOMETRIC FUNCTIONS OF (−θ)

$\sin(-\theta) = -\sin\theta$

cos(−θ) = cosθ

tan(−θ) = −tanθ

2. CIRCULAR FUNCTIONS OF ALLIED ANGLES

When θ is an acute angle, 90° − θ is called the angle **complementary** to θ. When θ is acute, θ and 180° − θ are called **supplementary** angles.

TABLE OF FORMULAE FOR ALLIED ANGLES

	180° − θ	**180° + θ**	**360° − θ**	**− θ**	**90° − θ**	**90° + θ**	**270° − θ**	**270° + θ**
Sin	sinθ	− sinθ	− sinθ	− sinθ	cosθ	cosθ	− cosθ	− cosθ

Cos	$-\cos\theta$	$-\cos\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\sin\theta$
Tan	$-\tan\theta$	$+\tan\theta$	$-\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$\cot\theta$	$-\cot\theta$

3. SOME IMPORTANT FACTS

(i) For any power *n*, $(\sin A)^n$ is written as $\sin^n A$. Similarly for all trigonometric ratios.

(ii) $\sin^2 A + \cos^2 A = 1$; $1 + \tan^2 A = \sec^2 A$ and $1 + \cot^2 A = \mathrm{cosec}^2 A$.

(iii) $|\sin A| \le 1 \Rightarrow -1 \le \sin A \le 1$

$|\cos A| \le 1 \Rightarrow -1 \le \cos A \le 1$

(iv) The trigonometric ratios are also called as trigonometric functions. They are also sometimes called circular functions.

ANGLE / RATIO	0°	30°	45°	60°	90°
sine	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cosine	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0
tangent	0	$1/\sqrt{3}$	1	$\sqrt{3}$	undefined
cotangent	undefined	$\sqrt{3}$	1	$1/\sqrt{3}$	0
secant	1	$2/\sqrt{3}$	$\sqrt{2}$	2	undefined
cosecant	undefined	2	$\sqrt{2}$	$2/\sqrt{3}$	1

***Example*:** **Evaluate: cos(−3030°).**

Solution : cos(−3030°) = cos(3030°) (using cos(−θ) = cosθ) = cos(8 × 360° + 150°)

$$= \cos 150° = \cos(180° - 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}.$$

***Example*:** **Find the values of the other five trigonometric functions if $\cos\theta = -\frac{1}{2}$.**

Solution : ∵ cos θ is negative in IInd and IIIrd quadrant only

In IInd quadrant sine and cosecant is positive and other trigonometric ratios are negative.

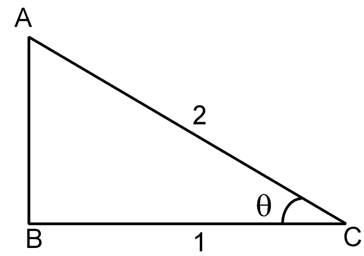
Now construct a right angle triangle with base *BC* = 1 and hypotenuse *AC* = 2

⇒ perpendicular $AB = \sqrt{3}$

$\Rightarrow \sin\theta = \frac{AB}{AC} = \frac{\sqrt{3}}{2}$ $\Rightarrow \text{cosec}\,\theta = \frac{AC}{AB} = \frac{2}{\sqrt{3}}$

$\Rightarrow \tan\theta = -\frac{AB}{BC} = -\sqrt{3}$ $\Rightarrow \cot\theta = -\frac{BC}{AB} = -\frac{1}{\sqrt{3}}$

and $\sec\theta = -\frac{AC}{BC} = -2$



In IIIrd quadrant only tangent and cotangent are positive and rest are negative

$\Rightarrow \sin\theta = -\frac{\sqrt{3}}{2}$, $\tan\theta = \sqrt{3},\ \cot\theta = \frac{1}{\sqrt{3}}$, $\sec\theta = -2$ and $\text{cosec}\,\theta = -\frac{2}{\sqrt{3}}$

Example: **Simplify:** $\frac{\sin\left(\frac{3\pi}{2}-\theta\right)\cos\left(\frac{\pi}{2}+\theta\right)}{\tan\left(\frac{\pi}{2}+\theta\right)} - \frac{\sin\left(\frac{3\pi}{2}-\theta\right)}{\sec(\pi+\theta)}$.

Solution : The expression = $\frac{(-\cos\theta)(-\sin\theta)}{-\cot\theta} - \frac{(-\cos\theta)}{(-\sec\theta)} = -\sin^2\theta - \cos^2\theta = -(\sin^2\theta + \cos^2\theta) = -1$.

Example : **Prove that $\text{cosec}^4\theta(1 - \cos^4\theta) = 1 + 2\cot^2\theta$.**

Solution : $\text{cosec}^4\theta(1 - \cos^4\theta) - 2\cot^2\theta$

$= \frac{\text{cosec}^2\theta(1-\cos^2\theta)(1+\cos^2\theta)}{\sin^2\theta} - 2\cot^2\theta$ $= \text{cosec}^2\theta(1 + \cos^2\theta) - 2\cot^2\theta$

$= \text{cosec}^2\theta + \cot^2\theta - 2\cot^2\theta = 1 + 2\cot^2\theta - 2\cot^2\theta = 1$

3. CIRCULAR FUNCTIONS OF COMPOUND ANGLES

1. ADDITION AND SUBTRACTION FORMULAE

- $\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$
- $\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$
- $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$
- $\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$
- $\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

- $\tan(\theta-\phi)=\dfrac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$
- $\cot(\theta+\phi)=\dfrac{\cot\theta\cot\phi-1}{\cot\phi+\cot\theta}$
- $\cot(\theta-\phi)=\dfrac{\cot\theta\cot\phi+1}{\cot\phi-\cot\theta}$

Example: **If sinθ = $\frac{8}{17}$ and cosβ = $\frac{9}{41}$, find sin(θ + β), cos(θ + β), sin(θ − β) and cos(θ − β), where θ is an obtuse angle and β is an acute angle.**

Solution : Since sinθ = $\frac{8}{17}$, $\cos^2\theta = 1-\frac{64}{289}=\frac{225}{289}$

∴ cosθ = $\pm\frac{15}{17}$. As θ is obtuse, cosθ is negative.

∴ cosθ = $-\frac{15}{17}$

$\sin^2\beta = 1-\frac{81}{1681}=\frac{1600}{1681}$

∴ sinβ = $\pm\frac{40}{41}$

As β is acute, sinβ is positive

∴ sinβ = $+\frac{40}{41}$

Now sin(θ + β) = sinθ cosβ + cosθ sinβ $=\frac{8}{17}\cdot\frac{9}{41}+\left(-\frac{15}{17}\right)\frac{40}{41}=\frac{72-600}{697}=-\frac{528}{697}$

cos(θ + β) = cosθ cosβ − sinθ sinβ $=\left(-\frac{15}{17}\right)\left(\frac{9}{41}\right)-\left(\frac{8}{17}\right)\left(\frac{40}{41}\right)=-\frac{455}{697}$

sin(θ − β) = sinθ cosβ − cosθ sinβ $=\left(\frac{8}{17}\right)\left(\frac{9}{41}\right)-\left(-\frac{15}{17}\right)\left(\frac{40}{41}\right)=\frac{672}{697}$

cos(θ − β) = cosθ cosβ + sin θ sinβ $=\left(-\frac{15}{17}\right)\left(\frac{9}{41}\right)+\left(\frac{8}{17}\right)\left(\frac{40}{41}\right)=\frac{185}{697}$.

Example: **Show that $\cos\theta+\sin(270^\circ+\theta)-\sin(270^\circ-\theta)+\cos(180^\circ+\theta)=0$.**

Solution : L.H.S. $=\cos\theta+\sin(270^\circ+\theta)-\sin(270^\circ-\theta)+\cos(180^\circ+\theta)$

$= \cos\theta - \cos\theta + \cos\theta - \cos\theta$

= 0 = R.H.S.

2. MULTIPLE ANGLE FORMULAE

(a) Functions of 2A

(i) $\sin 2A = 2\sin A \cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

(b) Functions of 3A

(i) $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

Example: **Prove that** $\dfrac{\tan(\pi/4 + A)}{\tan(\pi/4 - A)} = \dfrac{2\cos A + \sin A + \sin 3A}{2\cos A - \sin A - \sin 3A}$.

Solution : L.H.S. = $\dfrac{\dfrac{1+\tan A}{1-\tan A}}{\dfrac{1-\tan A}{1+\tan A}} = \dfrac{(1+\tan A)^2}{(1-\tan A)^2} = \dfrac{1+\tan^2 A + 2\tan A}{1+\tan^2 A - 2\tan A}$

$$= \frac{1+\dfrac{2\tan A}{1+\tan^2 A}}{1-\dfrac{2\tan A}{1+\tan^2 A}} = \frac{1+\sin 2A}{1-\sin 2A}$$

R.H.S. $= \dfrac{2\cos A + 2\sin 2A.\cos A}{2\cos A - 2\sin 2A\cos A} = \dfrac{2\cos A(1+\sin 2A)}{2\cos A(1-\sin 2A)} = \dfrac{1+\sin 2A}{1-\sin 2A}$

Both sides reduce to the same result.

3. EXPRESSING PRODUCTS OF TRIGONOMETRIC FUNCTIONS AS SUM OR DIFFERENCE

(i) $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

(ii) $2\cos A \sin B = \sin(A + B) - \sin(A - B)$

(iii) $2\cos A \cos B = \cos(A + B) + \cos(A - B)$

(iv) $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

Example: **Show that 8 sin 10° sin 50° sin 70° = 1.**

Solution : L.H.S. = 4 (2sin 50° sin 10°) sin70°

= 4 {cos(50° − 10°) − cos (50° + 10°)} sin70°, using 2 sin$A$ sin$B$

= 2{sin110° + sin30°} −2 sin70° since cos60° = 1/2.

= cos(A − B) − cos(A + B)

= 4(cos40° − cos60°) sin70°

= 2.(2sin70°. cos40°) − 4cos60° sin70°= 2sin70° + 2sin30° − 2sin70°

= 2sin30° = 1.

4. EXPRESSING SUM OR DIFFERENCE OF TWO SINES OR TWO COSINES AS A PRODUCT

$$\sin C + \sin D = 2\sin\frac{C+D}{2}.\cos\frac{C-D}{2}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}.\sin\frac{C-D}{2}$$

$\cos C + \cos D = 2\cos\frac{C+D}{2}.\cos\frac{C-D}{2}$

$\cos D - \cos C = 2\sin\frac{C+D}{2}.\sin\frac{C-D}{2}$

Example: **Show that** $\frac{\mathbf{\sin 7x - \sin 3x - \sin 5x + \sin x}}{\mathbf{\cos 7x + \cos 3x - \cos 5x - \cos x}} = \mathbf{\tan 2x}$.

Solution : Numerator = $(\sin 7x + \sin x) - (\sin 5x + \sin 3x)$

$= 2\sin 4x\ .\ \cos 3x - 2\sin 4x\ .\ \cos x$ {using C.D. formula}

$= 2\sin 4x\ (\cos 3x - \cos x)$

Denominator $= (\cos 3x - \cos 5x) - (\cos x - \cos 7x)$

$= 2\sin 4x \sin x - 2\sin 4x \sin 3x = 2\sin 4x\ (\sin x - \sin 3x)$

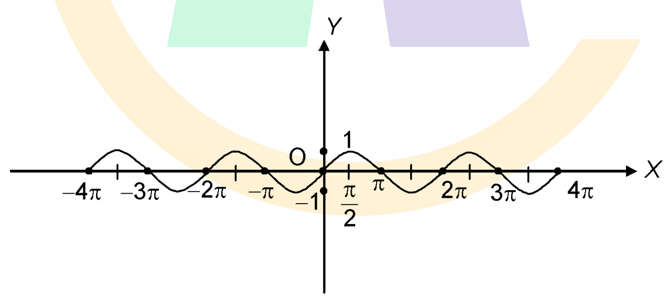
$\therefore$ the given expression $= \frac{\cos 3x - \cos x}{\sin x - \sin 3x} = \frac{\cos x - \cos 3x}{\sin 3x - \sin x} = \frac{2\sin 2x \sin x}{2\cos 2x \sin x} = \tan 2x$.

4. GRAPH OF TRIGONOMETRIC FUNCTIONS

1. **Graph of $y = \sin x$**

Domain : R

Range : $[-1, 1]$

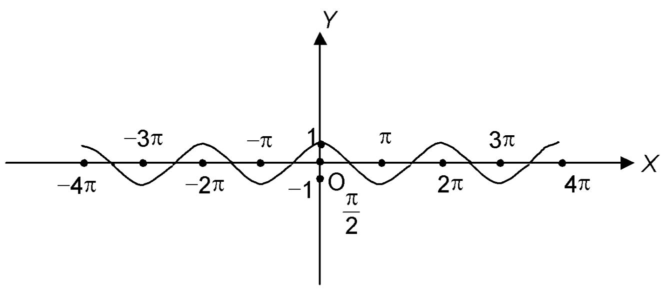


$y = \sin x$

2. **Graph of $y = \cos x$**

Domain : R

Range : $[-1, 1]$

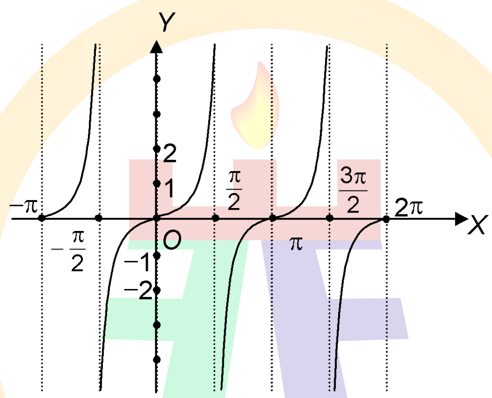


y = cosx

3. Graph of *y* = tan *x*

Domain : $R - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$

Range : *R*



y = tan x