LESSON-9

PERMUTATIONS AND COMBINATIONS

1. INTRODUCTION

A branch of mathematics where we count number of objects or number of ways of doing a particular job without actually counting them, is known as combinatorics and in this chapter we will deal with elementary combinatorics.

For example, if in a room there are five rows of chairs and each rows contains seven chairs, then without counting them we can say, total number of chairs is 35.

2. FUNDAMENTAL PRINCIPLE OF COUNTING

1. MULTIPLICATION PRINCIPLE OF COUNTING

If a job can be done in m ways, and when it is done in any one of these ways another job can be done in n, then both the jobs together can be done in mn ways. The rule can be extended to move number of jobs.

Example: **Find the number of three digit numbers in which all the digits are distinct, odd and number is a multiple of 5.**

Solution: Here it is equivalent to completing three jobs of filling units, tens and hundred place. Now number of ways of filling unit place is only one i.e. 5. Now, four odd digits are left, hence ten's place can be filled in four ways and hundred's place in three ways.

$\therefore$ number of required three digit natural numbers is $1 \times 4 \times 3 = 12$.

2. ADDITION PRINCIPLE OF COUNTING

If a job can be done in m ways and another job can be done in n ways then either of these jobs can be done in $m + n$ ways. The rule can be extended to more number of jobs.

Example: **How many three digit numbers xyz with $x < y$ and $z < y$ can be formed.**

Solution: Obviously $2 \le y \le 9$. If $y = k$, then x can take values from 1 to $k - 1$ and z can take values from 0 to $k - 1$.

Thus required number of numbers $= \sum_{k=2}^{9} (k-1)(k) = 240$.

3. PERMUTATIONS

Each of different arrangement which can be made by taking some or all of a number of things is called a permutation. It is assumed that

- the given things let us say there are n of them are all distinct, that is, no two are alike.
- the arrangement is one of placing one thing next to another as in a straight line; hence it is also known as linear permutation.
- In any arrangement any one thing is used only once. In other words, there is no repetition.

1. COUNTING FORMULAE FOR PERMUTATION

To find the value of ^nP_r

Suppose there are r blank spaces in a row and n different letters. The number of ways of filling up the blank spaces with n different letters is the number of ways of arranging n things r at a time, i.e. ^nP_r. It must be noted that each space has to be filled up with only one letter.

1	2	3		r

The first space can be filled in n ways. Having filled it, there are $n - 1$ letters left and therefore the second space can be filled in $n - 1$ ways. Hence the first two spaces can be filled in $n(n - 1)$ ways. When the first two spaces are filled, there are $n - 2$ letters left, so that the third space can be filled in $n - 2$ ways. Therefore the first three spaces can be filled in $n(n - 1)(n - 2)$ ways; proceeding like this, the r spaces can be filled in $n(n - 1)(n - 2) \ldots [n - (r - 1)]$ ways.

The number of permutations of n things taken r at a time is denoted as ^nP_r and its value is equal to

$$^nP_r = n(n-1)(n-2) \ldots\ldots (n-r+1)$$

$$= \frac{n!}{(n-r)!}$$ (using factorial notation $n! = n(n-1) \ldots.. 3.2.1.$) where $0 \le r \le n$.

In particular

- The number of permutations of n different things taken all at a time $= {}^nP_n = n!$
- $^nP_0 = 1$, $^nP_1 = n$ and $^nP_{n-1} = {}^nP_n = n!$
- $^nP_r = n\,({}^{n-1}P_{r-1})$ where $r = 1, 2, \ldots\ldots n$.

Example: **Show that** $^{n}P_r = {}^{(n-1)}P_r + r \cdot {}^{(n-1)}P_{(r-1)}$ **.**

Solution: $^{(n-1)}P_r + r \cdot {}^{(n-1)}P_{(r-1)} = \dfrac{(n-1)!}{(n-1-r)!} + \dfrac{r(n-1)!}{(n-r)!}$

$$= (n-1)!\left[\frac{1}{(n-1-r)!} + \frac{r}{(n-r)!}\right]$$

$$= (n-1)!\,\frac{(n-r+r)}{(n-r)!} \qquad (\because (n-r)! = (n-r)\,(n-r-1)\,!)$$

$$= \frac{n(n-1)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

A common sense interpretation of the identity above is possible. The number of permutations of r things which may be made from n things, $r\,^{(n-1)}P_{r-1}$ contain one specified thing and $^{(n-1)}P_r$ do not contain that specified thing and these two together give $^{n}P_r$.

2. IMPORTANT RESULTS

Number of permutations under certain conditions:

- Number of permutations of n different things, taken r at a time, when a particular thing is to be always included in each arrangement is $r\,^{n-1}P_{r-1}$.
- Number of permutations of n different things, taken r at a time, when m particular things are never taken in any arrangement is $^{n-m}P_r$.
- Number of permutations of n different things, taken all at a time, when m specified things always come together is $(m\,!)\,(n - m + 1)\,!$.
- Number of permutations of n different things, taken all at a time, when m specified things never come together is $n! - m!\,(n - m + 1)\,!$.

Example: **Find the number of permutations of *n* different things taken *r* at a time so that two particular things are always included and are together.**

Solution: The two things can be combined as one unit. The remaining $(n-2)$ things may be permuted $(r-2)$ at a time in $^{n-2}P_{r-2}$ ways. In each arrangement of these $(r-2)$ things, are created $(r-1)$ spaces in which the unit of two things can be placed. Further the two things in the unit may be interchanged. The number of permutations is

$$^{n-2}P_{r-2} \cdot (r-1) \cdot 2 = \frac{2(r-1)\,(n-2)!}{(n-r)!}$$

3. PERMUTATION OF *n* DISTINCT OBJECT WHEN REPETITION IS ALLOWED

- The number of permutations of n different things taken r at time when each thing may be repeated any number of times is n^r.

Example: **Show that the total number of permutations of n different things taken not more than 'r' at a time, each being allowed to repeat any number of times, is $n\dfrac{n^r-1}{(n-1)}$.**

Solution: Number of a permutations taken one at a time = n

Number of permutations taken two at a time = n^2

.. ..

Number of permutations taken r at a time = n^r

$\therefore$ Total number of permutations

$$= n + n^2 + n^3 + \ldots\ldots\ldots + n^r$$

$$= \frac{n(n^r - 1)}{(n-1)}$$

4. ARRANGEMENT OF *n* THINGS WHEN ALL ARE NOT DISTINCT

- If given n things are not all distinct, then it is possible that few many be of one kind, and few others may be of second kind, etc. In such case, the number of permutations of n things taken all at a time, where p are alike of one kind, q are alike of second kind and r are alike of third kind and the rest $n - (p + q + r)$ are all distinct is given by

$$\frac{n!}{p!\,q!\,r!} \quad (p + q + r \le n)$$

Example: **Find the number of ways in which we can arrange four letters of the word MATHEMATICS.**

Solution: The letters of the word MATHEMATICS are (M, M), (A, A), (T, T), H, E, I, C and S, making eight distinct letters. We can choose four out of them in ${}^8C_4$ = 70 ways, and arrange each of these sets of four in 4! = 24 ways, yielding (70) (24) = 1680 arrangements. Second, we can choose one pair from among the three identical letter pairs, and two distinct letters out of the remaining seven in (${}^3C_1$) (${}^7C_2$) = (3) [(7 × 6)/2] = 63 ways. The letters so obtained can be arranged in 4!/2! = 12 ways, so the number of arrangements in this case is (63) (12) = 756. Finally, we can choose two pairs out of the three identical letter pairs. This can be done in ${}^3C_2$ = 3 ways and the letters obtained can be arranged in 4!/2!2! = 6 ways, so that the number of arrangements in this last case is (3) (6) = 18. Hence the total number of arrangements is 1680 + 756 + 18 = 2454.

4. COMBINATIONS

Each of different grouping or selections that can be made by some or all of a number of given things without considering the order in which things are placed in each group, is called combinations.

1. COUNTING FORMULAE FOR COMBINATIONS

The number of combinations (selections or groupings) that can be formed from n different objects taken r at a time is denoted by ${}^nC_r$ and its value is equal to

$$ {}^nC_r = \frac{n!}{(n-r)!\ r!} \quad (0 \le r \le n)$$

as $$ {}^nC_r = \frac{{}^nP_r}{r!}$$

as in a permutation the arrangement of r selected objects out of n, is done in $r!$ ways and in combination arrangement in a group is not considered.

In particular

- ${}^nC_0 = {}^nC_n = 1$ i.e. there is only one way to select none or to select all objects out of n distinct objects.
- ${}^nC_1 = n$ There are n ways to select one thing out of n distinct things.
- ${}^nC_r = {}^nC_{n-r}$

  Therefore ${}^nC_x = {}^nC_y \Leftrightarrow\ x = y$ or $x + y = n$.
- If n is odd then the greatest value of ${}^nC_r$ is ${}^nC_{\frac{n+1}{2}}$ or ${}^nC_{\frac{n-1}{2}}$.
- If n is even then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$.

Example: **Prove that product of *r* consecutive positive integer is divisible by *r* ! .**

Solution: Let r consecutive positive integers be (m), $(m + 1)$, $(m + 2)$, ….. , $(m + r - 1)$, where $m \in \mathbf{N}$.

$\therefore$ Product $= m(m + 1)\,(m + 2)\,\ldots..\,(m + r - 1)$

$$= \frac{(m-1)!\ m(m+1)\,(m+2).....(m+r-1)}{(m-1)!}$$

$$= \frac{(m+r-1)!}{(m-1)!} = r!\,.\,\frac{(m+r-1)!}{r!\,(m-1)!}$$

which is divisible by r ! (∵ ${}^{m+r-1}C_r$ is natural number)

Example: **Show that the number of triangles whose angular points are at the vertices of a given polygon of n sides but none of whose sides are the sides of the polygon is $\dfrac{n\,(n-4)\,(n-5)}{6}$.**

Solution: For any triangle to be possible, 3 of the n vertices are to be chosen. This can be done in ${}^nC_3$ ways. Of these there are n triangles with two sides as adjacent sides of the polygon – like side 1 and side 2; side 2 and side 3 etc, the third side of the triangle being the corresponding diagonal; and there are, with one side of the polygon as a side of the triangle, $(n-4)$ triangles.

$\therefore$ Required number of triangles $= {}^nC_3 - n - n\,(n-4)$

$$= \frac{n(n-1)(n-2)}{6} - n - n(n-4)$$

$$= \frac{n}{6}\{n^2 - 3n + 2 - 6 - 6n + 24\}$$

$$= \frac{n}{6}(n^2 - 9n + 20) = \frac{n(n-4)(n-5)}{6}$$

2. IMPORTANT RESULTS OF COMBINATIONS (SELECTIONS)

- The number of ways in which r objects can be selected from n distinct objects if a particular object is always included is ${}^{n-1}C_{r-1}$.
- The number of ways in which r objects can be selected from n distinct objects if m particular objects are always included is ${}^{n-m}C_{r-m}$.
- The number of ways in which r objects can be selected from n distinct objects if m particular objects are always excluded is ${}^{n-m}C_r$.
- The number of ways in which r objects can be selected from n objects if m particular objects are identical is $\sum_{r=0}^{r} {}^{n-m}C_r$ or $\sum_{r=r-m}^{r} {}^{n-m}C_r$ according as $r \le m$ or $r > m$.

Example: **A bag contains 23 balls in which 7 are identical. Then find the number of ways of selecting 12 balls from bag.**

Solution: Here $n = 23$, $p = 7$, $r = 12$ $(r > p)$

Hence, required number of selections $= \sum_{r=5}^{12} {}^{16}C_r$

$= {}^{16}C_5 + {}^{16}C_6 + {}^{16}C_7 + {}^{16}C_8 + {}^{16}C_9 + {}^{16}C_{10} + {}^{16}C_{11} + {}^{16}C_{12}$

$= ({}^{16}C_5 + {}^{16}C_6) + ({}^{16}C_7 + {}^{16}C_8) + ({}^{16}C_9 + {}^{16}C_{10}) + ({}^{16}C_{11} + {}^{16}C_{12})$

$= {}^{17}C_6 + {}^{17}C_8 + {}^{17}C_{10} + {}^{17}C_{12}$ $\quad (\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r)$

$= {}^{17}C_{11} + {}^{17}C_9 + {}^{17}C_{10} + {}^{17}C_{12}$ $\quad (\because {}^nC_r = {}^nC_{n-r})$

$= ({}^{17}C_{11} + {}^{17}C_{12}) + ({}^{17}C_9 + {}^{17}C_{10}) = {}^{18}C_{12} + {}^{18}C_{10} = {}^{18}C_6 + {}^{18}C_8$