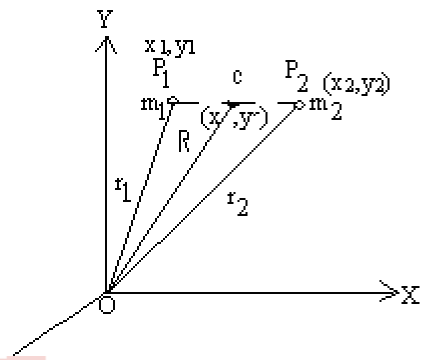
CENTRE OF MASS – SYSTEM OF PARTICLES

(i) A two Particles system

The centre of mass of system of particles is a fixed point at which the whole mass of the system may be considered to be concentrated.

Consider two particles P_1 and P_2 of masses m_1 and m_2 situated in a reference frame with O as its origin. Let r_1 and r_2 be their position vectors a shown in fig. The position vector R of the centre of mass C is

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$



Let $m_1 + m_2 = M$., Then

$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 \cdots\cdots\cdots\cdots\cdots\cdots(1)$$

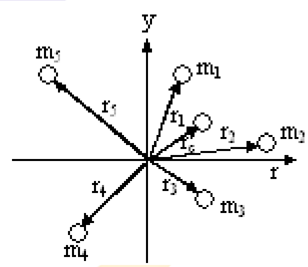
Let the co-ordinate of P_1 be (x_1, y_1); P_2 be (x_2,y_2) and C be

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{M}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2}{M}$$

(ii) Discrete System of n particles

For a system of n particles as shown in the position vector of the centre of mass is defined as the weighted average of the individual position vectors, that is



$$r_{cm} = \frac{m_1r_1 + m_2r_2 + \dots\dots + m_nr_n}{m_1 + m_2 + \dots\dots + m_n}$$

where $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ are the position vectors of masses m_1, m_2, …., m_n respectively

.or $r_i = \frac{\sum m_ir_i}{M}$ where $M = \sum m_i$

(iii)Continuous System

The centre of mass of a continuous body is defined as

$$\vec{r}_{cm} = \frac{1}{M}\int r\,dm$$

In the component form

$$x_{cm} = \frac{1}{M}\int x\,dm; \quad y_{cm} = \frac{1}{M}\int y\,dm; \quad z_{cm} = \frac{1}{M}\int z\,dm.$$

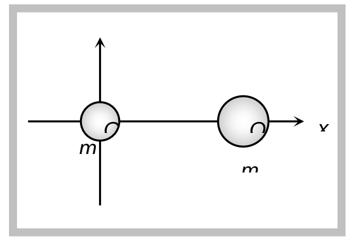
Q. Find Centre of mass of Rod ?

***Problem* 1.** The distance between the carbon atom and the oxygen atom in a carbon monoxide molecule is 1.1 Å. Given, mass of carbon atom is 12 *a.m.u.* and mass of oxygen atom is 16 *a.m.u.*, calculate the position of the center of mass of the carbon monoxide molecule

(a) 6.3 Å from the carbon atom (b)1 Å from the oxygen atom

(c) 0.63 Å from the carbon atom (d) 0.12 Å from the oxygen atom

Solution : (c) Let carbon atom is at the origin and the oxygen atom is placed at *x*-axis



$m_1 = 12$, $m_2 = 16$, $\vec{r}_1 = 0\hat{i} + 0\hat{j}$ and $\vec{r}_2 = 1.1\hat{i} + 0\hat{j}$

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{16 \times 1.1}{28}\hat{i}$$

$\vec{r} = 0.63\,\hat{i}$ *i.e.* 0.63 Å from carbon atom.

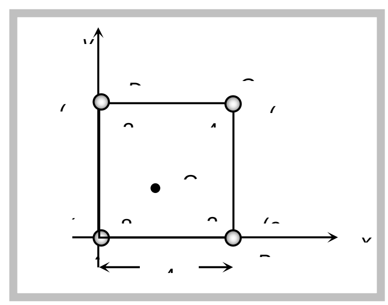
***Problem* 2.** Masses $8, 2, 4, 2\ kg$ are placed at the corners A, B, C, D respectively of a square $ABCD$ of diagonal $80\ cm$. The distance of centre of mass from A will be

(a) $20\ cm$ (b) $30\ cm$ (c) $40\ cm$ (d) $60\ cm$

Solution : (b) Let corner A of square *ABCD* is at the origin and the mass 8 *kg* is placed at this corner (given in problem) Diagonal of square $d = a\sqrt{2} = 80\ cm \Rightarrow a = 40\sqrt{2}cm$

$m_1 = 8kg,\ m_2 = 2kg,\ m_3 = 4kg,\ m_4 = 2kg$

Let $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$ are the position vectors of respective masses



$\vec{r}_1 = 0\hat{i} + 0\hat{j}$, $\vec{r}_2 = a\hat{i} + 0\hat{j}$, $\vec{r}_3 = a\hat{i} + a\hat{j}$, $\vec{r}_4 = 0\hat{i} + a\hat{j}$

From the formula of centre of mass

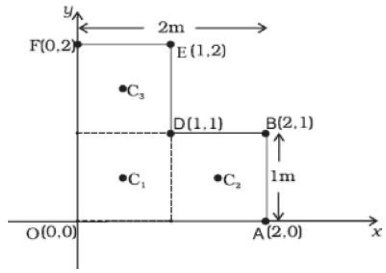
$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4}{m_1 + m_2 + m_3 + m_4} = 15\sqrt{2}\hat{i} + 15\sqrt{2}\hat{j}$$

$\therefore$ co-ordinates of centre of mass $= (15\sqrt{2}, 15\sqrt{2})$ and co-ordination of the corner $= (0, 0)$

From the formula of distance between two points (x_1, y_1) and (x_2, y_2) distance

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(15\sqrt{2} - 0)^2 + (15\sqrt{2} - 0)^2} = \sqrt{900} = 30\,cm$$

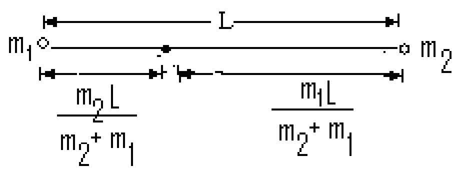
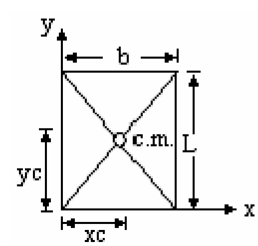
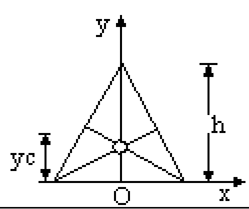
Example. 1 Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimensions as shown below. The mass of the lamina is 3 kg.



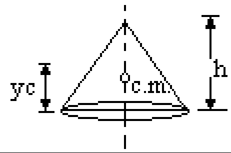
Example 2.Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100g, 150g, and 200g respectively. Each side of the equilateral triangle is 0.5m long.

***NCERT* 7. 2.** In the HC1 molecule, the separation between the nuclei of the two atoms is about 1.27 Å (1 Å = 10^{-10} m). Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.

. **(iv) Centre of mass of some common system.**

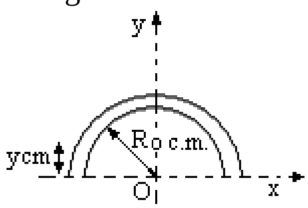
▪ A system of two point masses. The center of mass lies closer to the heavier mass. 
• **A rectangular Plate** $x_c = \frac{b}{2}$ $y_c = \frac{L}{2}$ 
▪ A Triangular Plate ▪ $y_c = \frac{h}{3}$ 

- A Circular $y_c = \frac{h}{3}$



- A semi-circular ring

$y_c = \frac{2R}{\pi}$

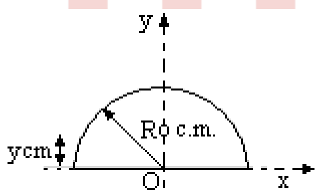


$x_c = 0$

- A Semi-circular ring

$y_c = \frac{4R}{3\pi}$

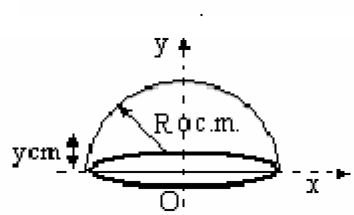
$x_c = 0$



- **A Hemispherical Disc**

$y_c = \frac{R}{2}$

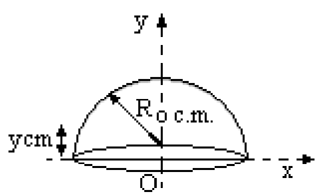
$x_c = 0$



- A Solid Hemisphere

$y_c = \frac{3R}{8}$

$x_c = 0$



(iv)Motion of the Centre of Mass

The velocity of centre of mass $\vec{V}_{cm} = \frac{d\vec{R}}{dt}$

$\vec{v}_1 = \frac{d\vec{r}_1}{dt}$ and $\vec{v}_2 = \frac{d\vec{r}_2}{dt}$

From eqn. (1) $M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2$

$$\frac{Md\vec{R}}{dt} = \frac{m_1 d\vec{r}_1}{dt} + \frac{m_2 d\vec{r}_2}{dt}$$

$$M\vec{V}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 \ldots\ldots\ldots\ldots\ldots(2)$$

RHS of the equation is the total linear momentum of the system. This will be a constant, if no external force acts on the system then

$M\vec{V}_{cm} = a\ constant \quad or$

$\vec{V}_{cm} = a\ constant$

(v) Acceleration

The acceleration of centre of mass is the rate of change of velocity of centre of mass $\vec{V}_{cm}$ i.e. $\vec{a}_{cm} = \frac{d}{dt}\left(\vec{V}_{cm}\right)$

$\vec{a}_1 = \frac{d\vec{v}_1}{dt}; \quad .\vec{a}_2 = \frac{d\vec{v}_2}{dt}$

Differentiating eqn. (2)

$$\frac{Md\vec{V}_{cm}}{dt} = m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt}$$

$$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 \ldots\ldots\ldots(3)$$

$\vec{a}_1, \vec{a}_2$ acceleration of particles.

$m\vec{a}_1 + m\vec{a}_2$ gives the total force acting on the system $\vec{F}_{tot}$

$$\vec{F}_{tot} = \vec{F}_{ext} + \vec{F}_{internal} = \vec{F}_{ext} + \vec{F}_{12} + \vec{F}_{21} = \vec{F}_{ext}$$

Because $\vec{F}_{internal} = \vec{F}_{12} + \vec{F}_{21} = 0$, by Newton's 3rd law the force exerted on (1) by (2) $\vec{F}_{12}$ will always be equal and opposite to the force exerted on (2) by (1) i.e., $\vec{F}_{12} = -\vec{F}_{21}$.

Then firm eqn, (3) $M\vec{a}_{cm} = \vec{F}_{ext}$

The acceleration of the C.M. is due to external force. Thus the C.M. of the system moves as if a particle of mass equal to total mass of the system and subjected to the external force applied to the system.

(vi) Momentum Conservation and Centre of Mass Motion.

Consider a system of particles of total mass M. The net external force acting on the system is $\vec{F}_{ext}$. The acceleration of centre of mass is a_{cm}.

Then $Ma_{cm} = \vec{F}_{ext}$

If $\vec{F}_{ext} = 0$ then, $M\vec{a}_{cm} = 0, \vec{a}_{cm} = 0; \frac{d\vec{V}_{cm}}{dt} = 0$

i.e., $\vec{V}_{cm} = a\ constant$

Thus in the absence of external force, the velocity of the centre of mss of the system is a constant.

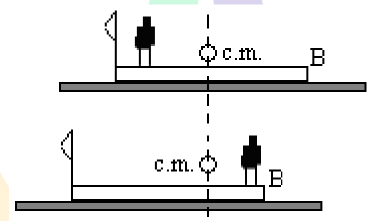
Equation (1) can also be written as

$M\vec{V}_{cm} = a\ constant$

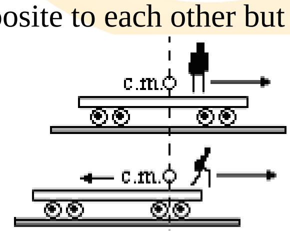
i.e., $m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + ... + m_N\vec{v}_N = a\ costant.$

The total linear momentum of a system of particle is a constant, if no external force acts on the system. This is the law of conservation of linear momentum.

In the fig. a man is standing at rest at the end A of the plank, which rests on a smooth horizontal surface. As the man walks to the other end B of the plank, the plank moves backward but the centre of mass of the system does not move because the net force on the system (mass + plank) is zero.



A man standing at the edge of a stationary trolley . Suddenly he jumps off. The man and the trolley move exactly opposite to each other but the centre of mass remains stationary.



In the fig.(A), a block is gently placed on a plank, which is moving on a smooth horizontal surface. The friction force between the block and the plank accelerate the block and decelerates the plank but being an internal force it is unable to accelerate the centre of mass of the system (block + plank). The final common velocity of block and plank is the velocity of centre of mass.

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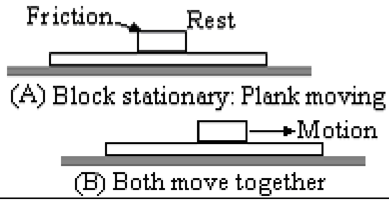
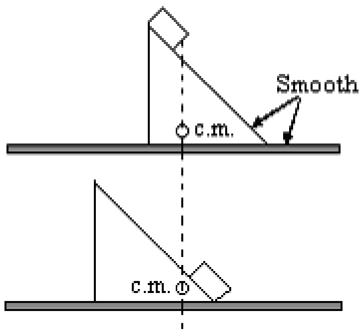
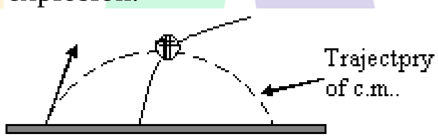


A block is kept at the top of a smooth wedge which in turn is kept on a smooth horizontal surface. As the block slides down the wedge, the wedge moves leftwards. Horizontally, the centre of mass does not move as there is no net horizontal force acting on the system (block + wedge). The centre of mass moves vertically downward as the net force (gravity) is acting downward.



A projectile fired form the ground explodes at its highest position into two fragments. Since explosion is an internal mechanism, therefore, it cannot affect the centre of mass of the system. The trajectory followed by the centre of mass after explosion is the same as it would have been without explosion.



MOMENTUM AND ENERGY CONSERVATION FOR A SYSTEM OF PARTICLES

In the absence of external impulses, the momentum of a system is conserved, i.e.

$$\Delta p = p_f - p_i = \int F_{ext} dt = 0$$

or $p_f = p_i$

In other words, velocity of the centre of mass remains constant.

In an inertial frame, in the absence of dissipative forces the mechanical energy of a system is conserved. That is

$$\Delta K + \Delta U = 0$$

Since, we can write K = K_c + K′, therefore

$$\Delta(Kc + K') + \Delta U = 0$$

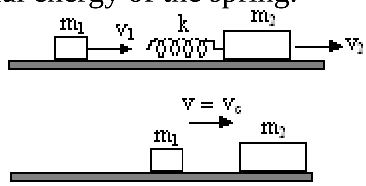
In the absence of net external force v_c = constant, thus

$$K_c = \frac{1}{2} M v_c^2 = \text{constant}$$

$$\Delta K' + \Delta U = 0$$

That is, in such a situation the kinetic energy of the system with respect to centre of mass may be converted into potential energy or vice-verse so that (K′ + U) remains constant.

(i)In fig two blocks m_1 and m_2 are moving with velocity v_1 and v_2 ($v_1 > v_2$) on a smooth horizontal floor. At the rear face of m_2 of spring of stiffness constant k is attached. As the approaching block m_1 touches the spring it stats acceleration of m_2 and retardation of m_1.The maximum compression occurs when both the blocks start moving toghter which is the velocity of the centre of mass v_c. since $F_{ext} = 0$, therefore, v_c = constant and $K_c = \frac{1}{2}Mv_c^2$ = constant. Hence, the kinetic energy K′ with respect to the centre of mass is Converted into the potential energy of the spring.

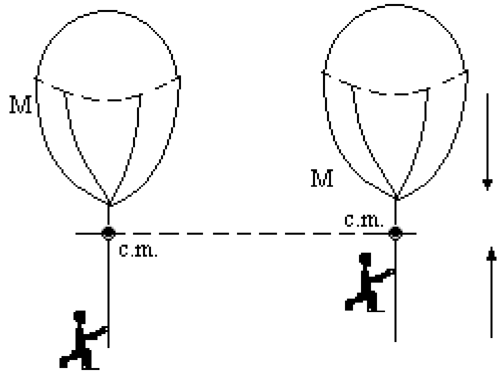


(ii) A small block of mass m is projected horizontally with a velocity v_o on a smooth wedge whose surface smoothly curves from horizontal to vertical as shown in fig. The wedge, in turn, can move freely on sooth horizontal surface. As the block and the wedge accelerates the block vertically but decelerates horizontally. When the block reaches the higher position its vertical velocity become zero, but horizontally the block and the wedge both move together with the velocity of centre of mass. Once again, $K_c = \frac{1}{2}(M+m)v_c^2$ remains constant; and K′ gets converted into the potential energy of the block.

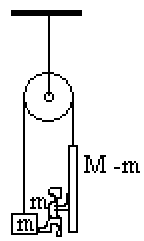
Misconception

- It is a common misconception that for a system in the vertical plane the centre of mass always accelerate downward as there is always a net force gravity acting on the system in the vertically downward direction.
- One must remember that the centre of mass accelerates under the action of the net force not under the action of gravity. If a system in the vertical plane is in equilibrium then centre of mass does not accelerate even if the gravity is acting.

A man of mass m is suspended in air by holding the rope of a balloon of mass M. The system (balloon + man) is in the equilibrium under the action of gravity and buoyant force. The centre of mass is stationary. As the man climbs up the rope the balloon moves downward but the centre of mass of the system remains stationary.



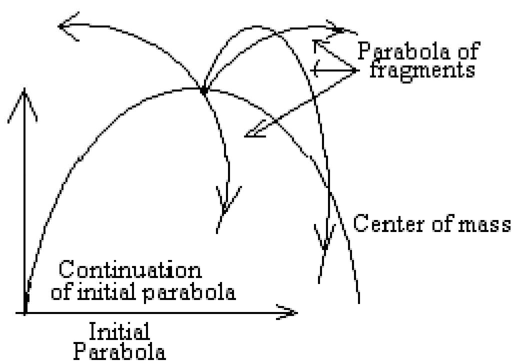
In the fig. a heavy block of mass m balances the weight of ladder plus man on the other side.
What happens to the centre of mass of the system as the man starts climbing up?
Wrong: The centre of mass does not shift because the system is in equilibrium.
Correct: The centre of mass shift upward. As the man moves upward a net force acts on the system in the upward direction.



Some Example of Centre of Mass Motion.

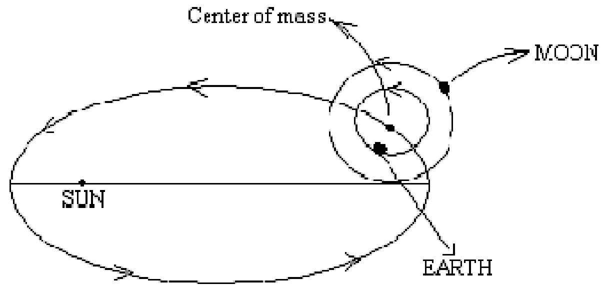
1. Explosion of a Cracker or a Shell

Initially the cracker is moving along a parabolic path. While in flight it explodes. Each fragment will follow its own parabolic path. But the centre of mass of all the fragments will continue to be along the same parabolic path of the shell as before, because there was no external force is acting on it. Explosion is caused by internal force only.



2. Motion of Earth-moon System

Moon goes round the sun in a circular orbit. The earth goes round the sun in elliptical orbit. It is more correct to say that earth and the moon both move in circular orbits about their common centre of mass describes an elliptical orbit round the sun. Here earth and moon form a system and their gravitational force of attraction is an internal fore. But sun's attraction on both is external force acting on their centre of mass as shown.



(vii) Centre of Mass of a Rigid Body

The centre of mass a rigid body is a fixed point with respect to the body as a whole. Centre of mass depends on (i) the geometrical shape of the body and (ii) the distribution of mass. Centre of mass is a point that represents the average location for the total mass of the system..

(viii) Important Points of Centre of Mass

- The centre of mass of a system is an imaginary point where the whole mass of the system may supposed to be concentrated at that point.
- The centre of mass of a body may lie inside or outside the body.
- The centre of mass of a body always lies on the axis of symmetry of the body if it exists. For a body in which there are two or more axes of symmetry, then the centre of mass lies at their at their point of intersection.
- In a system of n particles, the centre of mass of the system may or may not coincide with any of the particles.
- In a system of n coplanar particles the centre of mass of the system must lie within or at the edge of at least one of the polygons formed by joining (n – 1) particles.

(ix) Centre of gravity and Centre of mass

Centre of mass is a point where the whole mass (m) of the body is supposed to be concentrated. Centre of gravity is a point where the whole weight (mg) is supposed to be concentrated. The two points are identical if g is same at different parts of the body. For ordinary sized bodies C.G. and C.M. coincide. But for extended bodies for which g is not same at the different parts of the body the two points will not coincide.

Centre of mass.

1. Three point masses of 1kg, 2kg and 3kg lie at (1,2), (0, -1) and (2, -3) respectively. Calculate the co-ordinates of the centre of mass of the system.

2 .Two particles of masses 1 kg are located at $(2\hat{i}+5\hat{j}+13\hat{k})$ *and* $(-6i+4\hat{j}-2\hat{k})$ metre respectively. Find the position of their centre of mass.

3. In the HCl molecule, the separation between the nuclei of two atoms is about $1.27\overset{o}{A}\ (1\overset{o}{A}=10^{-10}m).$ Find the approximate location of the C.M. of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.

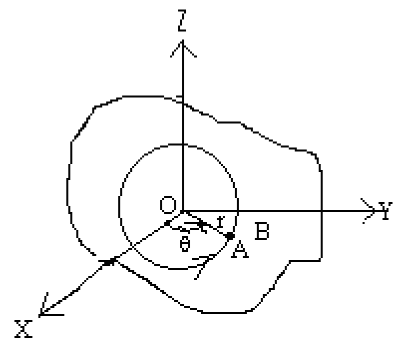
4. Three masses 3, 4,and 5kg are placed at the vertices of an equilateral triangle of 1m sides. Locate centre of mass.

5. Two particles of masses 100 gm and 300 gm have at a given time, position $2\hat{i}+5\hat{j}+13\hat{k}$ and $-6\hat{i}+4\hat{j}-2\hat{k}$ respectively and velocity $10\hat{i}-7\hat{j}-3\hat{k}$ **and** $-7\hat{i}-9\hat{j}-6\hat{k}$ m/s respectively. Deduce the instantaneous position and velocity of the centre of mass.

ROTATIONAL MOTION

Angular Displacement

Here θ is angular displacement. It is a vector quantity, which directed along the z-axis. The direction can be found out by using the right hand screw rule.

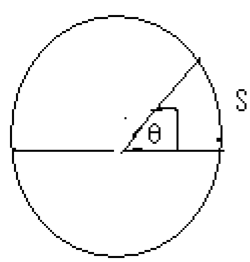


Angular velocity

The rate of change of angular displacement with time is called angular velocity.

$$\vec{\omega}=\frac{\Delta\theta}{\Delta t}$$

The instantaneous angular velocity $\omega = \lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}=\frac{d\theta}{dt}$. S.I. unit of ω is rad/s.



since $s = r\,\theta$

$$\frac{ds}{dt}=r\frac{d\theta}{dt}$$

$\mathbf{v} = r\,w$

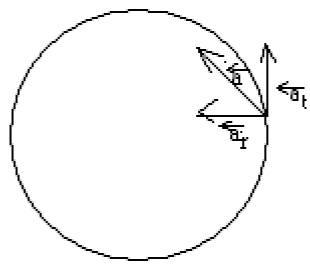
Linear speed = radius x angular velocity

Angular Acceleration

The rate of change of angular velocity with time is called angular acceleration. Let the instaneous angular speed of a rigid body change from ω_1 and ω_2 in a interval Δt. The change in angular velocity $\Delta\omega = \omega_2 - \omega_1$. Average angular acceleration is $\vec{\alpha} = \frac{\Delta\omega}{\Delta t}$

The instantaneous angular acceleration is given by $\alpha = \underset{\Delta t \to 0}{Lt} \frac{d\omega}{dt}$.

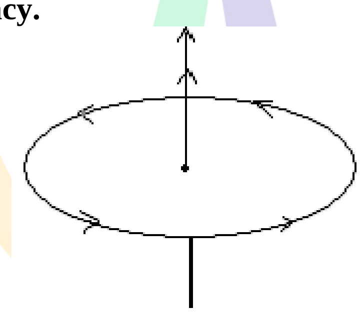
S.I unit of angular acceleration is rad/s^2. Angular acceleration is a vector, the direction is same as that of $\vec{\omega}$.



The net acceleration is the resultant of a_t and a_r. Both acceleration will be present if the circular motion is non-uniform.

Angular velocity in terms of Frequency and Period

Time taken by the body to complete one rotation is called **period.** Number of rotation made per second is called **frequency.**



We know $\omega = \frac{\theta}{t} = \frac{2\pi}{T}$

But frequency $n = \frac{1}{T}$. $So \quad \omega = 2\pi n.$

ω is also called angular frequency.

Equations of Rotational Motion

Consider a body moving along a straight line with an initial velocity u, having a uniform accelerations a. Let its final velocity be v after a time t and the displacement during the time be s. Then the equation for translatory motion are

(i) $v = u + at \quad (ii)\ s = ut + 1/2at^2 \quad (iii)\ v^2 = u^2 + 2as \quad (iv)\ S_n = u + \frac{1}{2}a(2n-1)$

For a rigid body rotating with uniform angular acceleration about an axis, we can derive similar equations

$(i)\ \omega = \omega_0 + \alpha t \quad (ii)\ \theta = \omega_0 t + \frac{1}{2}at^2 \quad (iii)\ \omega^2 = \omega_0{}^2 + 2\alpha\theta \quad (iv)\ \theta_{nth} = \omega_0 + \frac{1}{2}\alpha(2n-1)$

Derivation

(i) Angular velocity attained by a uniformly accelerating rigid after an interval of time.

Consider a rigid body rotating about an axis with an angular velocity ω_0. Let its uniform angular acceleration be α and its final angular velocity be ω after a time t.

The instantaneous angular acceleration $\alpha = \frac{d\omega}{dt}; \quad d\omega = \alpha\, dt$

$$\int d\omega = \int \alpha\, dt = \alpha \int dt$$

When t = 0, $\omega = \omega_0$ *and when* $t = t., \omega = \omega$

$$\int d\omega = \int \alpha\, dt = \alpha \int dt ,$$

$$\int_{w0}^{w} dw = \alpha \int_0^t dt$$

$w - w_o = \alpha t$,

$w = w_o + \alpha t$

(ii) **Angle covered during an interval of time**

The instantaneous angular velocity $\omega = \frac{d\theta}{dt}; d\theta = \omega dt = (\omega_0 + \alpha t)dt$

Let the angular displacement be 0, when t = 0 and at time t, the angular displacement be θ..

$$\int_0^\theta d\theta = \int_0^t (\omega_0 + \alpha t) dt = \left[\omega_0 t + \frac{\alpha t^2}{2}\right]_0^t$$

$$\theta - 0 = \omega_0 t + \frac{\alpha t^2}{2}$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

θ can be also be expressed in terms of the average angular velocity ω_{av}.

$$\omega_{av} = \frac{\omega + \omega_0}{2}$$

Let us assume the average angular velocity is equal to the uniform angular velocity for a time interval t, then the angular displacement $\theta = \omega_{av} x t$

$$\theta = \left(\frac{\omega_0 + \omega}{2}\right)t$$

(iii) Angular velocity attained after a certain time.

Instantaneous angular acceleration $\alpha = \frac{d\omega}{dt}$

$$\alpha = \frac{d\omega}{d\theta}.\frac{d\theta}{dt} = \frac{d\omega}{d\theta}.\omega$$

Let the angular velocity be ω_0 when $\theta = 0$ and let angular be ω when $\theta = 0$

$$\int_0^\theta d\theta = \int_{\omega_0}^{\omega} \omega d\omega; \alpha[\theta]_0^\theta = \left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega}$$

$$\alpha\theta = \frac{\omega^2 - \omega_0^2}{2}; 2\alpha\theta = \omega^2 - \omega_0^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

(iv) Angle covered in the second

The instantaneous angular velocity $\omega = \frac{d\theta}{dt}$

$$d\theta = \omega dt = (\omega_0 + \alpha t) = \omega_0 dt + \alpha t dt$$

Let the angle covered in the nth second be $\theta_n th$. This can be found out by integrating the above expression within the limits (n – 1) and n i.e., at time n, the angular displacement is θ_n and at time (n – 1), the angular displacement is θ_{n-1}

$$\int_0^\theta d\theta = \int_{n-1}^n \omega_0 dt + \alpha \int_{n-1}^n t dt$$

$$[\theta]_{\theta_{n-1}}^{\theta_n} = \omega_0 [t]_{n-1}^n + \alpha \left[\frac{t^2}{2}\right]_{n-1}^n$$

$$\theta_n - \theta_{n-1} = \omega_0 [n - (n-1)] + \alpha \left[\frac{n^2 - (n-1)^2}{2}\right]$$

$$\theta_n th = \omega_0 + \frac{1}{2}\alpha(2n-1)$$

***Problem* 3.** The wheel of a car is rotating at the rate of 1200 revolutions per *minute*. On pressing the accelerator for 10 *sec* it starts rotating at 4500 revolutions per *minute*. The angular acceleration of the wheel is

(a) 30 *radians/sec*^{2} (b) 1880 *degrees/sec*^{2} (c) 40 *radians/sec*^{2} (d) 1980 *degrees/sec*^{2}

Solution: (d) Angular acceleration (α) = rate of change of angular speed

$$= \frac{2\pi(n_2 - n_1)}{t} = \frac{2\pi\left(\frac{4500 - 1200}{60}\right)}{10} = \frac{2\pi \frac{3300}{60}}{10} \times \frac{360}{2\pi} \frac{degree}{sec^2} = 1980 \ degree / sec^2.$$

***Problem* 4.** Angular displacement (θ) of a flywheel varies with time as $\theta = at + bt^2 + ct^3$ then angular acceleration is given by **[BHU 2000]**

(a) $a + 2bt - 3ct^2$ (b) $2b - 6t$ (c) $a + 2b - 6t$ (d) $2b + 6ct$

Solution: (d) Angular acceleration $\alpha = \frac{d^2\theta}{dt^2} = \frac{d^2}{dt^2}(at + bt^2 + ct^3) = 2b + 6ct$

***Problem* 5.** A wheel is at rest. Its angular velocity increases uniformly and becomes 60 *rad*/*sec* after 5 *sec*. The total angular displacement is

(a) 600 rad (b) 75 rad (c) 300 rad (d) 150 rad

Solution: (d) Angular acceleration $\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{60-0}{5} = 12 rad/sec^2$

Now from $\theta = \omega_1 t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2}(12)(5)^2 = 150\ rad.$

***Problem* 6.** If the position vector of a particle is $\vec{r} = (3\hat{i} + 4\hat{j})$ meter and its angular velocity is $\vec{\omega} = (\hat{j} + 2\hat{k})$ *rad/sec* then its linear velocity is (in *m/s*)

(a) $(8\hat{i} - 6\hat{j} + 3\hat{k})$ (b) $(3\hat{i} + 6\hat{j} + 8\hat{k})$ (c) $-(3\hat{i} + 6\hat{j} + 6\hat{k})$ (d) $(6\hat{i} + 8\hat{j} + 3\hat{k})$

Solution: (a) $\vec{v} = \vec{\omega} \times \vec{r} = (3\hat{i} + 4\hat{j} + 0\hat{k}) \times (0\hat{i} + \hat{j} + 2\hat{k})$ $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 8\hat{i} - 6\hat{j} + 3\hat{k}$

Problem 7. The second hand of a clock slows down to 50 revolutions in 60s. So the number o revolutions made per second is 1/60 rps.
i.e., n_0 = 1/60 rps = 0.017 rps
Final frequency n = $\frac{50}{60x60} = 0.014 rps$
Time t = 7 minutes = 7 x60 = 420s
Acceleration $\alpha = ?; \omega = \omega_0 + \alpha t$

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{2\pi n - 2\pi n_0}{t} = \frac{2\pi(0.014 - 0.017)}{420}$$
$$= -4.48x10^{-5} rad/s^2$$

NCERT .7.14 .The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds.
(i) What is its angular acceleration, assuming the acceleration to be uniform?
(ii) How many revolutions does the engine make during this time?

Moment of a Force or Torque

When an external force acting on a body has a tendency to rotate the body about a fixed point or about a fixed axis, it is said to exert a **torque** on the body.
The moment of a force or the torque due to a force gives us the turning effect of the force about the fixed point/axis. It is measured by the product of magnitude of force and perpendicular distance of the line of action of force from the axis of rotation. Torque is represented by **τ**. Thus

Moment of force or Torque= force x perpendicular distance

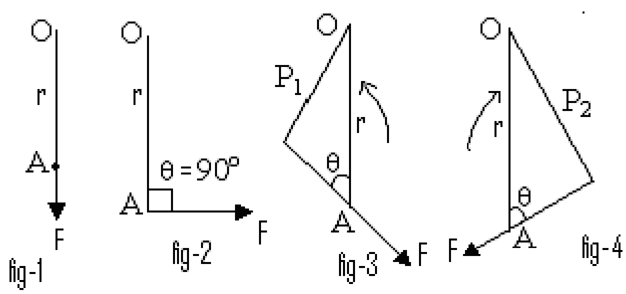
We shall prove that

$\vec{\tau} = \vec{r} x \vec{F} = r\,F\sin\theta\,\hat{n}$

where θ is smaller angular displacement between $\vec{r}$ and $\vec{F}$.$\hat{n}$ is unit vector along $\vec{\tau}$. The direction of $\vec{\tau}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{F}$, and is determined by right-handed screw rule. The S.I.unit of torque is N-m, which is equivalent to joule. The dimensions of torque are $\left[M^1L^2T^{-2}\right]$.

To understand the moment of force further, let us consider a rod OA of length r, suspended form O and capable of rotation about O. A force F is applied at the free end A of the rod. In angle between $\vec{r}$ and $\vec{F}$ is θ = 0º. Therefore, magnitude of moment of force

Or torque



$\tau = rF\sin\theta = rF\sin 0^{o} = 0$ as shown in fig-1

i.e. turning effect of the force in this case is zero.

In fig.-2 $\theta = 90^{o}$

$\therefore \quad \tau = rF\sin\theta = rF\sin 90^{o} = rF = \max imum$

i.e. turning effect of the force in this case, is maximum.

In fig.-3

$\tau = rF\sin\theta = F(r\sin\theta) = FxP_1$

where $P_1 = r\sin\theta =$perpendicular distance of line of action of force O. The rotation is anticlockwise.

In fig.-4

$\tau = rF\sin\theta = F(r\sin\theta) = FxP_2$

where $P_2 = r\sin\theta =$perpendicular distance of line of action of force from O. The rotation is of line of action of force from O. The rotation is clockwise.

BY convention**,** *anticlockwise moments are taken as positive and clockwise moments are taken as negative.*

Expression for Torque in Cartesian Co-ordinates form Rotation of a Particle in a Plane: Physical Meaning of Torque

The expression for torque can be obtained when we calculate work done by a forced in rotating a particle of mass m in a plane.

Consider a particle of mass m rotating in plane XY about the origin O. Let P the position of the particle at any instant, where $\overrightarrow{OP} = \vec{r}$ and ∠ XOP = θ. Let the rotation occur under the action of a force $\vec{F}$ applied at P, along $\overrightarrow{PA}$, fig.5

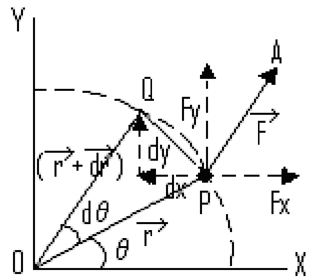


Fig-5

In a small time dt, let the particle at P reach Q, where $\overrightarrow{OQ} = \left(\vec{r} + \overrightarrow{dr}\right)$ *and* ∠POQ = dθ.

In vector triangle OPQ, $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$

or $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (\vec{r} + \overrightarrow{dr}) - \vec{r}$

$\overrightarrow{PQ} = \overrightarrow{dr}$

Small amount of work done in rotating the particle form P to Q is

dW = $\vec{F}.\overrightarrow{dr}$

If F_x, F_y are rectangular components of force $\vec{F}$ and dx, dy are rectangular components of displacement $\overrightarrow{dr}$, then

$\vec{F} = (\hat{i}F_x + \hat{j}F_y)$

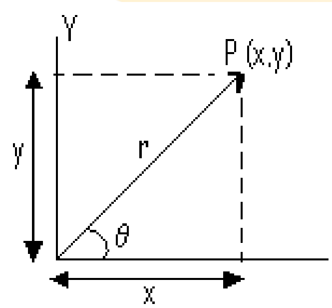
$d\vec{r} = \left(\hat{i}dx + \hat{j}dy\right)$

we get,

$dW = (\hat{i}F_x + \hat{j}F_y).\left(\hat{i}dx + \hat{j}dy\right)$

$dW = F_x dx + F_y dy$ ………(a)

FIG-6



Let the co-ordniates of the point P be (x,y)

As is clear from fig 6

$x = r\cos\theta$ ….(i)

Differentiating (i) with respect to θ.

$\frac{dx}{d\theta} = \frac{d}{d\theta}(r\cos\theta) = r\frac{d}{d\theta}(\cos\theta)$

$= r(-\sin\theta)$

$= -r\sin\theta = -y$ …..(iii)

$so, dx = -y\,d\theta$ ……(iv)

Again, difference (ii) and (iv)

$\frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta) = r\cos\theta = x$ …..(v)

dy = x dθ…………………..(v)

Substituting in (a) , we get

$dW = F_x(-yd\theta) + F_y(xd\theta)$

$= xF_y d\theta - yF_x d\theta$

dW = $(xF_y - yF_x)d\theta$

where $\tau = (xF_y - yF_x)$………….(b)

. Eqn (b) is the expression for torque in Cartesian co-ordinates.

We define torque τ *as a quantity in rotational motion, which when multiplied by a small angular displacement gives us work in rotational motion. This quantity corresponds to force in linear motion, which when multiplied by a small linear displacement gives us work done in linear motion. This is the physical meaning of torque.*

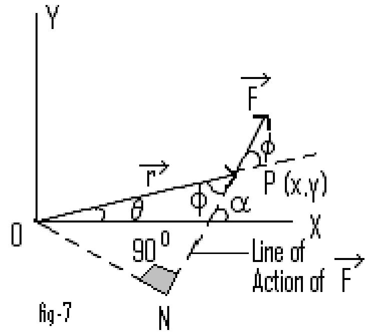
Expression for Torque In Polar Co-ordinates

Suppose the line for action of force $\vec{F}$ makes an angle α with X-axis, Fig,

$F_x = F\cos\alpha$, $F_y = F\sin\alpha$

If x,y are the co-ordinates of the point P, where $\overrightarrow{OP} = \vec{r}$ and ∠XOP = θ,

then x = r cos θ , y = r sin θ



Substituting these values in eqn (b), we get

$\tau = (r\cos\theta)F\sin\alpha - (r\sin\theta)(F\cos\alpha)$

$= rF[\sin\alpha\cos\theta - \cos\alpha\sin\theta]$

$\tau = rF\sin(\alpha - \theta)$

Let φ be the angle which the line of action of $\vec{F}$ makes with the position vector $\overrightarrow{OP} = \vec{r}$.

$\theta + \varphi = \alpha$ or $\varphi = \alpha - \theta$

$\tau = rF\sin\varphi$

This equation shows that ability of $\vec{F}$ to rotate the body depends not only on the magnitude of $\vec{F}$, but also on just how far from O, the force is applied.
From, draw ON ⊥ on the line of action of $\vec{F}$.
In ΔOPN, $\sin\varphi = \frac{ON}{OP} = \frac{ON}{r}$

$\therefore \quad ON = r\sin\phi$

$$\tau = r\ F\sin\phi = F(r\sin\phi) = F(ON) \ldots\ldots\text{(c)}$$

Hence torque due to a force is the product of force and perpendicular distance of line of action of force from the axis of rotation.
From ,eqn.(c) torque due to a force is maximum, when r is maximum. For example, we can open or close a door easily by applying force near the edge of the door (at maximum distance from the hinges). That is why a handle/knob is provided near the free edge of the plank of the door.
Similarly, to unscrew a nut fitted tightly to a bolt, we need a wrench with a long arm,
When the length of arm r is long, the force (F) required to produce a given turning effect (r x F) is smaller.

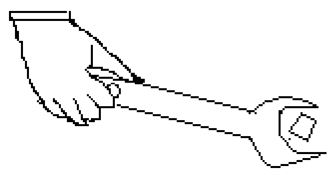


fig-8

(ii) Torque will be maximum, when $\sin\varphi$ = max. = +1. Therefore, $\varphi = 90^o$, i.e. when force is applied in a direction perpendicular to $\vec{r}$. For example, it is easiest to open or close a door by applying force at the edge of the plank in a direction perpendicular to the plank of the door.
(iii) When $\varphi = 0^o$ or 180^o,
$\sin\varphi = \sin 0^o$ or $\sin 180^o = 0$. Therefore,
$\tau = rF\sin\phi = 0$
i.e. torque of the force is zero. For example, the door cannot be rotated by applying force in a direction parallel to the plank of the door.
(iv) Eqn, can be rewritten in vector form as
$\vec{\tau} = \vec{r} x \vec{F}$
Obviously, torque is a vector quantity whose direction is given by right handed screw rule. Fig. shows relative orientation of $\vec{r}$ and $\vec{F}$. Force f acting actually at P has been shifted to origin O, in a direction parallel to itself.

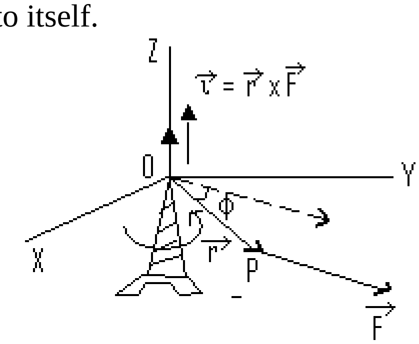


fig-9

When $\vec{F}$ rotates the particle in AY plane in anticlockwise direction, the tip of the right handed screw moves along the positive Z direction, which is the direction of the torque τ.

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Power Associated with Torque

work done (dW) in rotating a particle through a small angle (dW) in rotating a particle through a small angle (dθ) as

$dW = \tau\,(d\theta)$

If this work is done in a small time interval dt, we get

$$\frac{dW}{dt} = \tau\left(\frac{d\theta}{dt}\right)$$

Now, by definition, $\frac{dW}{dt} = P,$ the average power associated with torque,

and $\frac{d\theta}{dt} = \omega$, average angular speed of the body in this interval.

From, $P = \tau\omega$

i.e. power associated with torque is given by the product of torque and angular speed of the body about the axis of rotation.

In linear motion, the corresponding relation for power is $P = \tau\,\omega$

The Concept of Angular Momentum

Expression for Angular Momentum in Cartesian Co-ordinates

We know the concept of torque as the rotating effect or turning effect of a force. Torque is the rate of change of angular momentum..

If $P_x = mv_x$ and $P_y = mv_y$ are the x and y components of linear momentum of the body, then According to Newton's 2^{nd} law of motion,

$$F_x = \frac{dp_x}{dt} = \frac{d}{dt}(mv_x) = m\frac{dv_x}{dt}$$

and $$F_y = \frac{dp_x}{dt} = \frac{d}{dt}(mv_y) = m\frac{dv_y}{dt}$$

Substituting in eqn. (b) In, we get

$$\tau = x\,m\frac{dv_y}{dt} - y\,m\frac{dv_x}{dt}$$

$$\tau = m\left[x\frac{dv_y}{dt} - y\frac{dv_x}{dt}\right] \qquad \ldots.(i)$$

Using rule for differentiation of products, we may write

$$\frac{d}{dt}(xv_y - yv_x)$$

$$= x\frac{dv_y}{dt} + v_y\frac{dx}{dt} - y\frac{dv_x}{dt} - v_x\frac{dy}{dt}$$

$$= x\frac{dv_y}{dt} + v_yv_x - y\frac{dv_x}{dt} - v_xv_y$$

$$\left[\because\ \frac{dx}{dt} = v_x;\frac{dy}{dt} = v_y\right]$$

$= x\frac{dv_y}{dt} - y\frac{dv_x}{dt}$ ....(ii)

Substituting (ii) in (i), we obtain

$\tau = m\frac{d}{dt}(xv_y - yv_x)$

$\tau = \frac{d}{dt}(xmv_y - ymv_x)$

As $\tau = mv_y = p_y \quad and \quad mv_x = p_x$

$\therefore \quad \tau = \frac{d}{dt}(xp_y - yp_x)$

Put $(xp_y - yp_x) = L$ ....(iii)

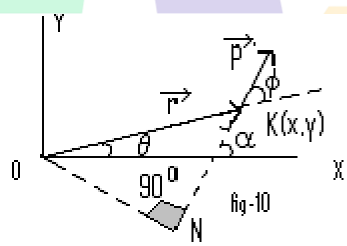
$\therefore \quad \tau = \frac{dL}{dt}$ ....(d)

Thus we have obtained torque τ as the rate of change of a quantity L as angular momentum of the body.

Eqn. represents expression for angular momentum in Cartesian co-ordinates. Thus basically, angular momentum of a particle/ body about a given axis is the moment of linear momentum of the particle / body about that axis.

Expression for Angular Momentum in Polar Co-ordinates

Suppose K(x,y) is position of a particle of mass m and linear momentum $\vec{p}$ rotating in XY plane about Z-axis. Fig. Let $\overrightarrow{OP} = \vec{r}$ and ∠XOK = θ.



∴ x = r cos θ and y = r sin θ …(iv) fig-10

Let the line of action of linear momentum $\vec{p}$ make an angle α with OX and angle φ with $\vec{r}$.

∴ p_x = p cos α and p_y = p sin α.........(.v)

putting the value(iv) & (v) in eqn. (iii)

$L = (r\cos\theta)(p\sin\alpha) - (r\sin\theta)(p\cos\alpha)$ …..

$= pr[\sin\alpha\cos\theta - \cos\alpha\sin\theta]$

$L = pr\sin(\alpha - \theta)$

From fig.-10 , θ + φ = α

φ = α - θ

$L = pr\sin\phi$ ....(e)

This is the expression for angular momentum of a particle in polar-coordinates.

From O, ON perpendicular to the line of action of $\vec{p}$.

In $\Delta KNO, \sin\phi = \frac{ON}{OK} = \frac{ON}{r}$

$ON = r\sin\phi$

$L = p(ON)$

Hence angular momentum of a body about a given axis is the product of linear momentum and perpendicular distance of line of action of linear momentum vector from the axis of rotation.

This is the physical meaning of angular momentum.

In S.I., the units of angular momentum are (kg ms^{-1}) (m) = kg m^2s^{-1}.The dimensional formula for angular momentum would be $[M^1L^2T^{-1}]$.

$\vec{L} = \vec{r} x \vec{p}$ ………………..(f)

This indicates that angular momentum is a vector quantity and its direction is given by right handed screw rule. When the centre of mass rotates in XY plane, say from X-axis towards Y-axis, the angular momentum vector $\vec{L}$ is along OZ i.e. the positive Z-direction, as illustrated in fig.-11.

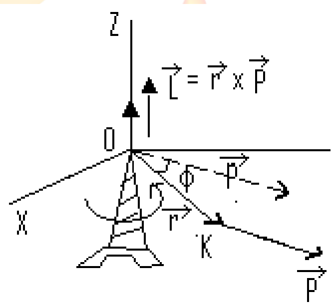


fig-11

Note.1. Proceeding as in the case of torque, we can show that radial component of linear momentum does not contribute to angular momentum of the particle. It is only the transverse component of linear momentum (perpendicular to position vector $\vec{r}$), which when multiplied by distance form the axis of rotation gives us angular momentum.

2. Again, proceeding as in case of torque, we may write the three rectangular rectangular components of angular momentum as

$L_x = y\,p_z - z\,p_y$, $L_y = z\,p_x - x\,p_z$, $L_z = x\,p_y - y\,p_x$

Geometrical Meaning of Angular Momentum

To understand the geometrical meaning of angular momentum, let us consider a particle rotating in XY plane about an axis OZ, fig. At any time t, let $\overrightarrow{OK} = \vec{r}$ be the position vector of particle

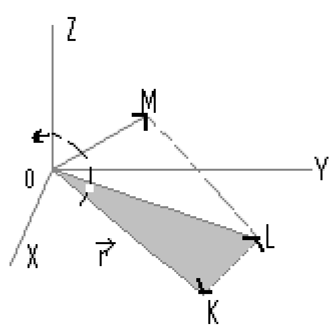


fig -12

In a small time dt, let the particle at K reach L, where $\overrightarrow{OL} = (\vec{r} + \overrightarrow{dr})$

Join $\overrightarrow{KL}$, which represents displacement of the particle in small tine dt.

In vector Δ OKL,

$$\overrightarrow{OK} + \overrightarrow{KL} = \overrightarrow{OL} \quad \text{and} \quad , \overrightarrow{KL} = \overrightarrow{OL} - \overrightarrow{OK}$$

$$= \left(\vec{r} + \overrightarrow{dr}\right) - \vec{r} = \overrightarrow{dr}$$

From O, draw $\overrightarrow{OM}$ equal and parallel to $\overrightarrow{KL}$. Join ML.

Area swept by the position vector in a small time dt is

$$\left|d\vec{A}\right| = \text{Area of } \Delta\text{OKL} = \frac{1}{2}\text{(area of parallelogram OKLM)}$$

$$= \frac{1}{2}\left|\overrightarrow{OK} x \overrightarrow{OM}\right| \qquad \text{So ,} \left|d\vec{A}\right| = \frac{1}{2}\left|\vec{r}\, x\, \overrightarrow{dr}\right|$$

Divide both sides by dt

$$\left|\frac{d\vec{A}}{dt}\right| = \frac{1}{2}\left|\vec{r}\, x \frac{\overrightarrow{dr}}{dt}\right| \qquad \text{But } \frac{\overrightarrow{dr}}{dt} = \vec{v} = \frac{\vec{p}}{m}$$

$$\therefore \left|\frac{d\vec{A}}{dt}\right| = \frac{1}{2m}\left|\vec{L}\right| \quad \left(\because \vec{r} x \vec{p} = \vec{L}\right)$$

or

$$\left|\vec{L}\right| = 2m\left|\frac{d\vec{A}}{dt}\right| \text{ ....(g)}$$

Here $\left|\frac{d\vec{A}}{dt}\right|$ is area swept by the position vector per unit time, and is called areal velocity of the position vector of particle.

Equation (g) shows that angular momentum of a particle about a given axis is twice the product of mass of the particle and areal velocity of position vector of the particle. This is the geometrical meaning of angular momentum.

Angular momentum is almost identical to the definition of linear momentum a the product of mass and linear velocity.

Problem **33.** A thin circular ring of mass *M* and radius *R* is rotating about its axis with a constant angular velocity ω. Four objects each of mass *m*, are kept gently to the opposite ends of two perpendicular diameters of the ring. The angular velocity of the ring will be

(a) $\frac{M\omega}{M+4m}$ (b) $\frac{(M+4m)\omega}{M}$ (c) $\frac{(M-4m)\omega}{M+4m}$ (d) $\frac{M\omega}{4m}$

Solution: (a) Initial angular momentum of ring $= I\omega = MR^2\omega$

If four object each of mass *m*, and kept gently to the opposite ends of two perpendicular diameters of the ring then final angular momentum = $(MR^2 + 4mR^2)\omega'$

By the conservation of angular momentum

Initial angular momentum = Final angular momentum

$MR^2\omega = (MR^2 + 4mR^2)\omega' \Rightarrow \omega' = \left(\frac{M}{M+4m}\right)\omega$.

Moment of Inertia (M.I.) of the Particle

M.I. is the rotational analogue of linear inertia. M.I. of a particle about an axis of rotation is given by the product of mass of the particle and the square of distance of the particle from the axis.

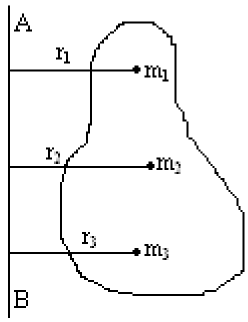
Moment of inertia $I = mr^2$, where m is the mass of the particle and r is the distance of the particle form the axis of rotation.

S.I. unit is kg m^2. Dimensional formula. $[ML^2]$

To find the M.I, of a body about an axis of rotation AB, we assume that the body is made up of a large number of particles of masses m_1, m_2, m_3 …. situated at distances r_1, r_2, r_3 …. from the axis of rotation.

$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^3 + \ldots\ldots$

$I = \Sigma mr^2$



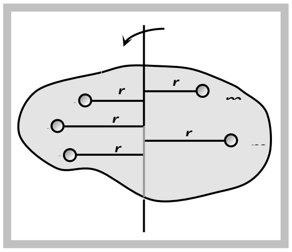
The M.I. of a body about an axis is defined as the sum of the products of the masses of different particles of the body and the square of their respective distances from the axis of rotation.

7.8 Radius of Gyration

Radius of gyration of a body about a given axis is the perpendicular distance of a point from the axis, where if whole mass of the body were concentrated, the body shall have the same moment of inertia as it has with the actual distribution of mass.

When square of radius of gyration is multiplied with the mass of the body gives the moment of inertia of the body about the given axis.

$I = Mk^2$ or $k = \sqrt{\frac{I}{M}}$.



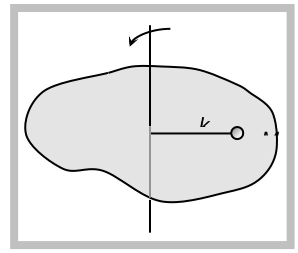
Here k is called radius of gyration.

From the formula of discrete distribution

$I = mr_1^2 + mr_2^2 + mr_3^2 + \ldots\ldots + mr_n^2$

If $m_1 = m_2 = m_3 = \ldots\ldots = m$ then

$$I = m(r_1^2 + r_2^2 + r_3^2 + \ldots\ldots\ldots r_n^2) \qquad \ldots\ldots\text{(i)}$$



From the definition of Radius of gyration,

$$I = Mk^2 \qquad \ldots\ldots\text{(ii)}$$

By equating (i) and (ii)

$$Mk^2 = m(r_1^2 + r_2^2 + r_3^2 + \ldots\ldots\ldots\ldots + r_n^2)$$

$$nmk^2 = m(r_1^2 + r_2^2 + r_3^2 + \ldots\ldots\ldots + r_n^2) \qquad [\text{As } M = nm]$$

$\therefore$

$$k = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \ldots\ldots\ldots + r_n^2}{n}}$$

Hence radius of gyration of a body about a given axis is equal to root mean square distance of the constituent particles of the body from the given axis.

(1) Radius of gyration (k) depends on shape and size of the body, position and configuration of the axis of rotation, distribution of mass of the body *w.r.t.* the axis of rotation.

(2) Radius of gyration (k) does not depends on the mass of body.

(3) Dimension $[M^0 L^1 T^0]$.

(4) S.I. unit : *Meter*.

(5) Significance of radius of gyration : Through this concept a real body (particularly irregular) is replaced by a point mass for dealing its rotational motion.

Example : In case of a disc rotating about an axis through its centre of mass and perpendicular to its plane

$$k = \sqrt{\frac{I}{M}} = \sqrt{\frac{(1/2)MR^2}{M}} = \frac{R}{\sqrt{2}}$$

So instead of disc we can assume a point mass M at a distance $(R/\sqrt{2})$ from the axis of rotation for dealing the rotational motion of the disc.

Note : ❑ For a given body inertia is constant whereas moment of inertia is variable.

Laws of Rotational Motion

First law. Every body continues in its state of rest or of uniform rotational motion about a fixed axis unless compelled by some external torque to change that state.

Second law. The time of change of angular momentum of a body about fixed axis is directly proportional to the applied torque and takes place in the direction of the torque applied.

Third law. The rate of charge of angular momentum of a body about a fixed axis is equal and opposite torque applied by the latter about the same axis of rotation.

Theorem of Moment of Inertia

1. Theorem of Parallel Axes

According to this theorem, the moment of inertia of a body about any axis is equal to the sum of moment of inertia about a parallel axis through its center of mass and the product of the mass of the body and the square of the distance between the two axes i.e.,
$I = I_{cm} + Ma^2$,

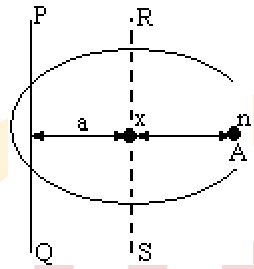
where I is the M.L. of the about any axis, Icm is the M.I. of the body about a parallel axis passing through its centre of mass, M mass of the body and a is the distance between the two axes.

Proof

Let PQ be the axis, about which the moment of inertia of the body is to be found out. RS is an axis parallel to PQ and at a distance **a** form it. Axis RS passes through the centre of mass of the body.

Let us consider a particle of a mass m at a distance x from RS.

Now moment of inertia of m about PQ = $m(x + a)^2$



∴ Moment of inertia of the whole body about PQ,

$I = \Sigma m\,(x + a)^2$

$I = \Sigma mx^2 + \Sigma ma^2 + 2\Sigma mxa$

If I_{cm} is the moment of inertia of the body about the axis RS passing through its centre of mass then

$I_{cm} = \Sigma mx^2$

Also $\sum ma^2 = a^2 \sum m = Ma^2$

Since the distance a between the two axes is constant and $\sum m = M$ is the mass of the body.

$I = I_{cm} + Ma^2 + 2a\Sigma mx$

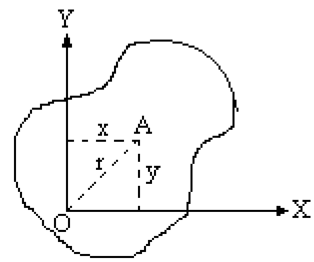
But $\sum mx$ is the sum of moments of all the particles about the axis RS which passes through its centre of mass. So the algebraic sum of all moments about it is zero i.e., $\Sigma mx = 0$.

Hence $\mathbf{I = I_{cm} + Ma^2}$

2. Theorem of Perpendicular Axes

This theorem states that the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of moment of inertia of lamina about two axes at right angles to each other, in its own plane, and intersecting each other at the point where the perpendicular axis passes through it.

If I_x and I_y be the moments of inertia of a plane lamina about OX and OY axes, which lie in the plane of the lamina and mutually perpendicular to each other, intersecting at the point O, then the moment of inertia I about as axis, which is passing through O and perpendicular to the plane of the lamina is given be



$I_z = I_x + I_y$

Proof

Let us consider a particle A of the lamina of mass m. The distance of the particle A from OX and OY axes are y and x respectively and the distance of A from O Z is r. Then, obviously

$I_z = \Sigma mr^2$

$I_x = \Sigma my^2$ and $I_y = \Sigma mx^2$

$I_x + I_y = \Sigma my^2 + \Sigma mx^2 = \Sigma m(y^2 + x^2) = \Sigma mr^2$ $[\therefore x^2 + y^2 = r^2]$

$\mathbf{I_x + I_y = I_z}$

Moments of Inertia of a Ring

(i) About an axis passing through its centre and perpendicular to its plane.

Consider M is mass of ring of radius R and O be its centre. We have to find the moment of inertia (M.I.) of the ring about and axis AB passing through its centre O and perpendicular to its plane.

Length of ring = $2\pi R$ = circumference

Mass per unit length $= M/2\pi R$

Consider a small element of the ring of length dx.

Mass of element $= \dfrac{M}{2\pi R} dx$

Moment of inertia of this element about axis AB

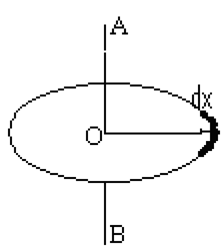
$dI = \left(\dfrac{M}{2\pi R} dx\right) \times R^2 = \dfrac{MR}{2\pi} dx$

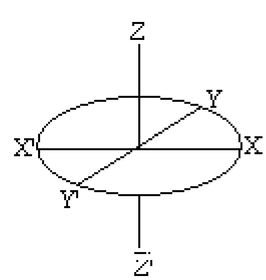
Moment of inertia of ring about axis AB

$$I = \int_0^{2\pi R} \frac{MR}{2\pi} dx = \frac{MR}{2\pi}[x]_0^{2\pi R} = \frac{MR}{2\pi}(2\pi R - 0) = MR^2$$

$\therefore$ $I = MR^2$

(ii) M.I. of a ring about its diameter. M.I. of the ring about any diameter will be the same due to its symmetrical shape. X′X and Y′Y are two diameters which are mutually perpendicular to each other.

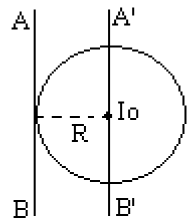




M.I. of the ring about the Z′Z axis = I = MR^2
Z′Z axis is perpendicular to both X′X and Y′Y. So perpendicular axes theorem can be used to find the M.I. of the ring about X′X Or Y′Y.
M.I. of the ring about Z′Z = M.I. about X′X + M.I. about Y′Y

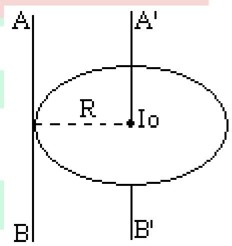
MR^2 = I + I = 2I, I = (1/2) MR^2

(iii) M.I. of a ring about a tangent lying in the plane of the ring
AB is a tangent to the ring lying in the plane of the ring. AB is parallel to the diameter A′B′ of the ring. So parallel axes theorem can be used to find the M.I. of the ring about AB (I).



$I = I_{cm} + Ma^2, Here \quad a = R, I_{cm} = MR^2/2, So, \quad I = MR^2/2 + MR^2 = (3/2)MR^2$

(iv) M.I. of a ring about a tangent perpendicular to the plane of the ring. AB is a tangent to ring, perpendicular to the plane of the ring. AB is parallel to A′B′ which is an axis passing through the centre of the ring and perpendicular to its plane. Using parallel axes theorem.



$I = I_{cm} + Ma^2, Here\ I_{cm} = MR^2, a = R$

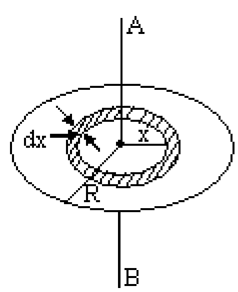
$I = MR^2 + MR^2 = 2MR^2$

M.I. of a Circular Disc

(i) **About an axis passing through the centre and perpendicular to its plane.**
Consider a disc of mass M and radius R.

Mass per unit area $= \dfrac{Mass}{Area}$

$\sigma = \dfrac{M}{\pi R^2}$



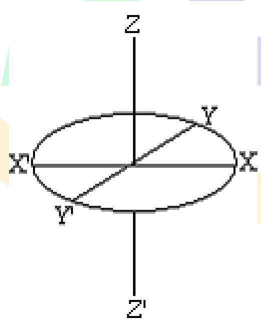
To find the M.I. of the disc about an axis AB passing through the centre and perpendicular to its plane, divide the disc into a number of concentric rings.
Let x be the radius and dx the radial thickness of one such elementary ring.
Area of the ring = 2πr x d*x*
Mass of the ring = Mass per unit area x Area of the ring
= σ x 2π*x dx*
M.I of ring about AB = σ 2π*x dx* x^2
dI = 2πσx^3dx
M.I of the whole disc I is equal to the sum of M.I. of all such concentric rings lying between x = 0 and x = R. This can be found out by integrating the above expression between the limit x = 0 and x = R

$$I = \int_0^R 2\pi\alpha^3 dx = 2\pi\sigma \int_0^R x^3 dx$$

$$= 2\pi\sigma \left[\frac{x^4}{4}\right]_0^R = 2\pi\sigma\left[\frac{R^4}{4} - 0\right] = \frac{2\pi\ M\ R^4}{4\ \pi\ R^2} = \frac{M\ R^2}{2}$$

(ii) **M.I. of a circular disc about a diameter. X′X and Y′Y are two diameters which are mutually perpendicular to each other.**
M.I. of the disc about by diameter will be the same, due to its symmetrical shape. By perpendicular axes theorem.
M.I. of the disc about Z′Z
= M.I. about X′X + M.I. about Y′Y

$$\frac{MR^2}{2} = I + I = 2I$$

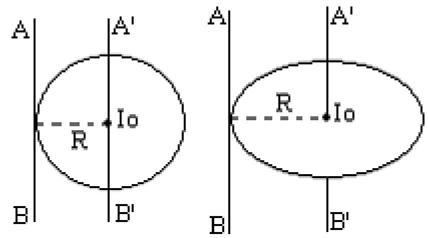


i.e. M.I. of the disc about any diameter , I $= \frac{1}{4}MR^2$

(iii) **M.I. of a disc about a tangent lying in the plane of disc**
AB is the tangent in the plane of disc. using parallel axes theorem.

$$I = I_{cm} + Ma^2, Here a = R, I_{cm} = MR^2/4$$

$$I = (MR^4/4) + MR^2 = (5/4)MR^2$$



(iv) **M.I. of a disc about a tangent perpendicular to the plane of the disc**

AB is the tangent perpendicular to the plane of the disc. using the parallel axes theorem.

$I = I_{cm} + Ma_2, herea = R, I_{cm} = MR^2 / 2)$

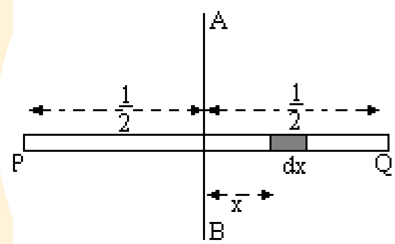
$I = (MR^2 / 2) + MR^2 = (3/2)MR^2$.

Moment of Inertia of a Rectangular Rod

(i) ***About an axis passing through the centre and perpendicular to its length***.

Consider a straight rod PQ of length l and mass M.

Mass per unit length of the rod = $\frac{mass}{l}$, $\rho = \frac{M}{l}$. We have to find the M.I. of the rod about the axis AB passing through the centre of the rod and perpendicular to its length. For this let us divide the rod into a number of infinitesimally small elements. Consider one such element of length dx at a distance x from O.



Mass of this element = ρ dx

M.I. of this element about the axis AB , dI= ρdx x^2

We have a number of such elements. so integrating between the limits $-\frac{l}{2} \quad to + \frac{1}{2}$

$$I = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \rho x^2 dx = 2\int_0^{\frac{1}{2}} \rho x^2 dx$$

$$= 2\rho \int_0^{\frac{1}{2}} x^2 dx = 2\rho \left[\frac{x^3}{3} \right]_0^{\frac{1}{2}}$$

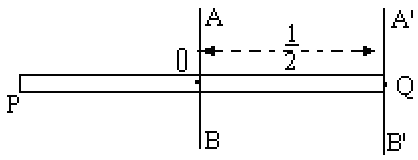
$$= \frac{2\rho}{3} \left[\frac{l^3}{8} - 0 \right] \quad But \rho = \frac{M}{l}$$

$$I = \frac{2}{3} \frac{M}{l} x \frac{l^3}{8} = \frac{Ml^2}{12}$$

(ii) **About an axis passing through one end of the rod and perpendicular to its length.**

AB and A′B′ are parallel axes. AB passes through the centre of mass of the rod. So parallel axes theorem can be applied, to find the M.I. of the rod about the axis A′B′ which passes through one of its ends.
M.I. of the rod about the axis A′B′ = I.
M.I. of the rod about the axis AB passing through the centre of mass of the rod

$$I_{cm} = \frac{Ml^2}{12}$$



Distance between the two axes = OQ = $\frac{l}{2}$.

I = $I_{cm} + Mx\left(\frac{1}{2}\right)^2$ (*By parallel axes theorem*)

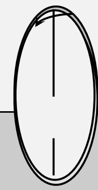
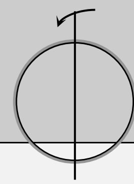
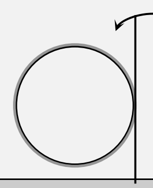
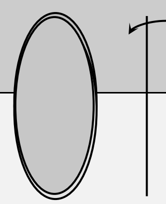
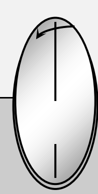
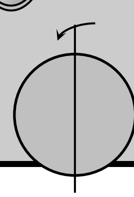
$$= \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{4Ml^2}{12}$$

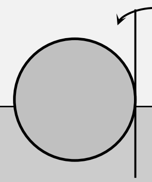
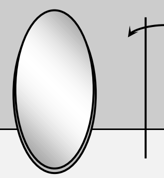
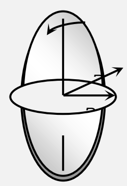
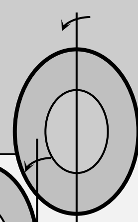
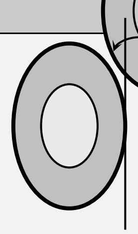
$$= \frac{Ml^2}{3}.$$

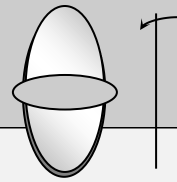
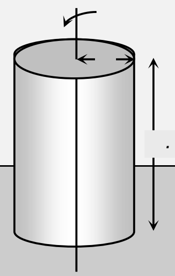
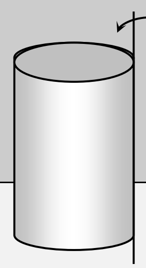
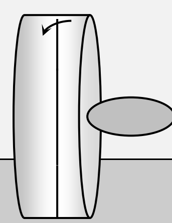
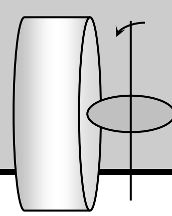
7.12 Analogy Between Tranlatory Motion and Rotational Motion

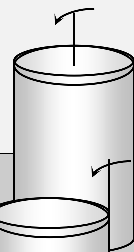
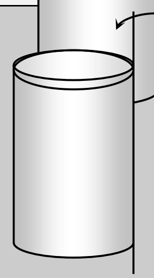
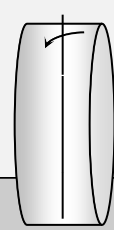
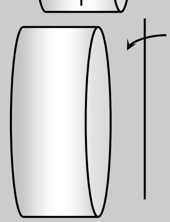
Translatory motion		Rotatory motion	
Mass	(m)	Moment of Inertia	(I)
Linear momentum	$P = mv$ $P = \sqrt{2mE}$	Angular Momentum	$L = I\omega$ $L = \sqrt{2IE}$
Force	$F = ma$	Torque	$\tau = I\alpha$
Kinetic energy	$E = \frac{1}{2}mv^2$ $E = \frac{P^2}{2m}$		$E = \frac{1}{2}I\omega^2$ $E = \frac{L^2}{2I}$

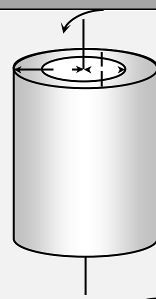
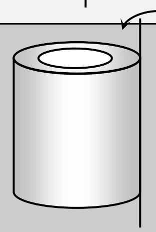
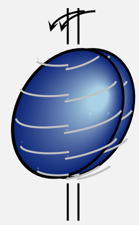
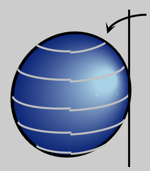
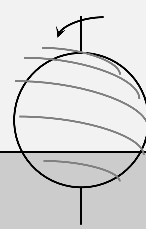
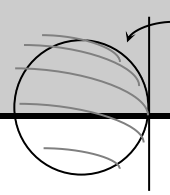
7.13 Moment of Inertia of Some Standard Bodies About Different Axes

Body	Axis of Rotation	Figure	Moment of inertia	*k*	k^2/R^2
Ring	About an axis passing through C.G. and perpendicular to its plane		MR^2	R	1
Ring	About its diameter		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$	$\frac{1}{2}$
Ring	About a tangential axis in its own plane		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$	$\frac{3}{2}$
Ring	About a tangential axis perpendicular to its own plane		$2MR^2$	$\sqrt{2}R$	2
Disc	About an axis passing through C.G. and perpendicular to its plane		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$	$\frac{1}{2}$
Disc	About its Diameter		$\frac{1}{4}MR^2$	$\frac{R}{2}$	$\frac{1}{4}$

Body	**Axis of Rotation**	**Figure**	**Moment of inertia**	***k***	k^2/R^2
Disc	About a tangential axis in its own plane		$\frac{5}{4}MR^2$	$\frac{\sqrt{5}}{2}R$	$\frac{5}{4}$
Disc	About a tangential axis perpendicular to its own plane		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$	$\frac{3}{2}$
Annular disc inner radius = R_1 and outer radius = R_2	Passing through the centre and perpendicular to the plane		$\frac{M}{2}[R_1^2 + R_2^2]$	–	–
Annular disc	Diameter		$\frac{M}{4}[R_1^2 + R_2^2]$	–	–
Annular disc	Tangential and Parallel to the diameter		$\frac{M}{4}[5R_1^2 + R_2^2]$	–	–

Body	**Axis of Rotation**	**Figure**	**Moment of inertia**	***k***	**k^2/R^2**
Annular disc	Tangential and perpendicular to the plane		$\frac{M}{2}[3R_1^2 + R_2^2]$	–	–
Solid cylinder	About its own axis		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$	$\frac{1}{2}$
Solid cylinder	Tangential (Generator)		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$	$\frac{3}{2}$
Solid cylinder	About an axis passing through its C.G. and perpendicular to its own axis		$M\left[\frac{L^2}{12} + \frac{R^2}{4}\right]$	$\sqrt{\frac{L^2}{12} + \frac{R^2}{4}}$	
Solid cylinder	About the diameter of one of faces of the cylinder		$M\left[\frac{L^2}{3} + \frac{R^2}{4}\right]$	$\sqrt{\frac{L^2}{3} + \frac{R^2}{4}}$	

Body	**Axis of Rotation**	**Figure**	**Moment of inertia**	***k***	k^2/R^2
Cylindrical shell	About its own axis		MR^2	R	1
Cylindrical shell	Tangential (Generator)		$2MR^2$	$\sqrt{2}R$	2
Cylindrical shell	About an axis passing through its C.G. and perpendicular to its own axis		$M\left[\frac{L^2}{12}+\frac{R^2}{2}\right]$	$\sqrt{\frac{L^2}{12}+\frac{R^2}{2}}$	
Cylindrical shell	About the diameter of one of faces of the cylinder		$M\left[\frac{L^2}{3}+\frac{R^2}{2}\right]$	$\sqrt{\frac{L^2}{3}+\frac{R^2}{2}}$	

Body	**Axis of Rotation**	**Figure**	**Moment of inertia**	***k***	k^2/R^2
Hollow cylinder with inner radius = R_1 and outer radius = R_2	Axis of cylinder		$\frac{M}{2}(R_1^2 + R_2^2)$		
Hollow cylinder with inner radius = R_1 and outer radius = R_2	Tangential		$\frac{M}{2}(R_1^2 + 3R_2^2)$		
Solid Sphere	About its diametric axis		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$	$\frac{2}{5}$
Solid sphere	About a tangential axis		$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}R$	$\frac{7}{5}$
Spherical shell	About its diametric axis		$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$	$\frac{2}{3}$
Spherical shell	About a tangential axis		$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}R$	$\frac{5}{3}$

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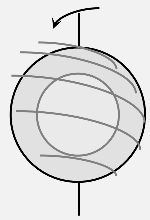
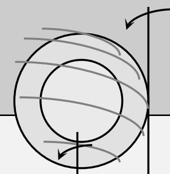
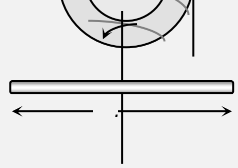
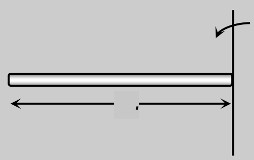
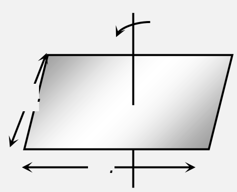
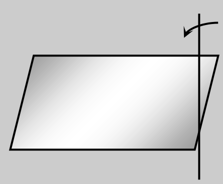
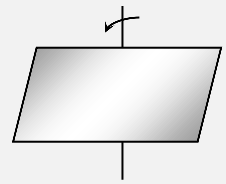
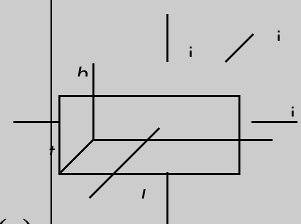
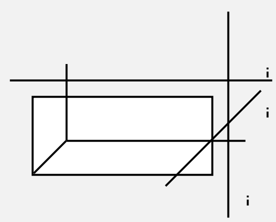
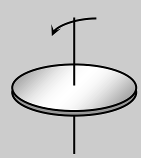
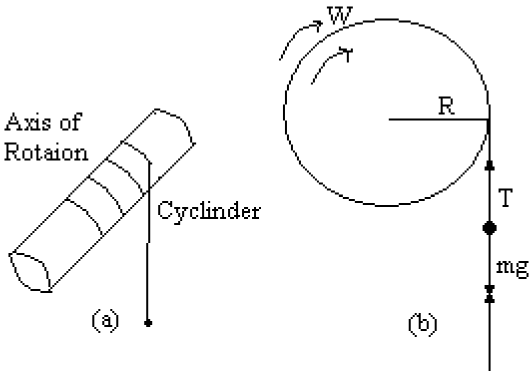
Body	Axis of Rotation	Figure	Moment of inertia	k	k^2/R^2
Hollow sphere of inner radius R_1 and outer radius R_2	About its diametric axis		$\frac{2}{5}M\left[\frac{R_2{}^5 - R_1{}^5}{R_2{}^3 - R_1{}^3}\right]$		
Hollow sphere	Tangential		$\frac{2M[R_2^5 - R_1^5]}{5(R_2^3 - R_1^3)} + MR_2^2$		
Long thin rod	About on axis passing through its centre of mass and perpendicular to the rod.		$\frac{ML^2}{12}$	$\frac{L}{\sqrt{12}}$	
Long thin rod	About an axis passing through its edge and perpendicular to the rod		$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$	
Rectangular lamina of length l and breadth b	Passing through the centre of mass and perpendicular to the plane		$\frac{M}{12}[l^2 + b^2]$		
Rectangular lamina	Tangential perpendicular to the plane and at the mid-point of breadth		$\frac{M}{12}[4l^2 + b^2]$		

Body	Axis of Rotation	Figure	Moment of inertia	*k*	k^2/R^2
Rectangular lamina	Tangential perpendicular to the plane and at the mid-point of length		$\frac{M}{12}[l^2+4b^2]$		
Rectangular parallelopiped length *l*, breadth *b*, thickness *t*	Passing through centre of mass and parallel to (i) Length (x) (ii) breadth (z) (iii) thickness (y)		(i) $\frac{M[b^2+t^2]}{12}$ (ii) $\frac{M[l^2+t^2]}{12}$ (iii) $\frac{M[b^2+l^2]}{12}$		
Rectangular parallelepiped length *l*, breath *b*, thickness *t*	Tangential and parallel to (i) length (x) (ii) breadth (y) (iii) thickness(z)		(i) $\frac{M}{12}[3l^2+b^2+t^2]$ (ii) $\frac{M}{12}[l^2+3b^2+t^2]$ (iii) $\frac{M}{12}[l^2+b^2+3t^2]$		
Elliptical disc of semimajor axis = *a* and semiminor axis = *b*	Passing through CM and perpendicular to the plane		$\frac{M}{4}[a^2+b^2]$		

A point mass on a string would on a cylinder with horizontal axis of rotation
Consider a solid cylinder of mass m and radius R. It is mounted symmetrically on a horizontal axle so that it can freely rotate about its axis. A mass less string is would round the cylinder and a mass m is suspended as shown in fig. When the mass is released from rest, it moves down with an acceleration **a.** Let T be the tension in the string.

The forces are (1) The weight of the body mg acting vertically downwards (2) The tension in the string T upwards .So



mg – T = ma ….(1)

But the tension in the string produces a torque in the cylinder τ = Iα

Also τ = Force x perpendicular distance = T x R

$1\alpha = TR, \alpha = \frac{TR}{I}$ ………..(2) But angular acceleration $\alpha = \frac{a}{R}, \ \alpha = \frac{TR}{I}$

i.e., $T = \frac{I\ a}{R^2}$ ………..(3)

Substituting the value of T in equation(1)

$$mg - \frac{I\ a}{R^2} = ma, mg = ma + \frac{Ia}{R}$$

$$mg = ma\left[1 + \frac{Ia}{maR^2}\right]$$

$$mg = ma\left[1 + \frac{Ia}{mR^2}\right]$$

$$a = \frac{g}{\left[1 + \frac{I}{mR^2}\right]}$$

Since I, m and R are position quantities, a is always less than g.

Substituting the value of a from equation (3) in equation

$$T = \frac{I}{R^2} x \frac{g}{\left[1 + \frac{I}{mR^2}\right]}$$

$$= \frac{I}{R^2} x \frac{g}{\left[\frac{I + m\ R^2}{mR^2}\right]} \qquad T = \frac{mg}{\frac{I + mR^2}{I}}$$

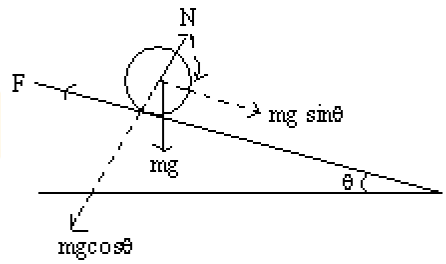
Thus T is always less than weight of the body

Cylinder rolling without slipping on an inclined plane.

Consider a solid cylinder initially at rest and rolling down a plane without slipping.
The force acting on a cylinder are

(1) mg acting vertically downward
(2) Normal reaction of the plane, N perpendicular to the plane upwards.
(3) The force of friction F acting parallel to the plane upwards. There is no acceleration in a direction normal to the plane. so, N = mg cosθ
Net downward force produces the acceleration **a.**
So mg sinθ - F = ma
The weight of the body and normal reaction are directed along the radius of the cylinder away form the centre. so they cannot produce a torque. The torque produced by force of friction is about the centre of mass.

$\tau = I\alpha$
$F.R = I\alpha$
But $I = \frac{1}{2}mR^2$, where I is the M.I. of cylinder about its axis of rotation



$$\alpha = \frac{a}{R}$$

$$F = \frac{1\alpha}{R} = \frac{1}{2}\frac{mR^2}{R} x \frac{a}{R} = \frac{1}{2}ma$$

Substituting the value of F in equation

$$mg\sin\theta - \frac{1}{2}ma = ma$$

$$\frac{3}{2}ma = mg\sin\theta$$

From equation

$$F = \frac{1}{2}ma = \frac{1}{2}mx\frac{2}{3}g\sin\theta$$

$$\frac{1}{3}ma\sin\theta$$

∴ Coefficient of static friction $\mu_s = \frac{F}{N}$

$$= \frac{1mg\sin\theta}{3mg\sin\theta} \qquad = \frac{1}{3}\tan\theta$$

Hence the condition to prevent slipping is $\mu_s \geq \frac{1}{3}\tan\theta$.

Problem based on moment of inertia

Problem **06.** A circular disc X of radius R is made from an iron plate of thickness t, and another disc Y of radius $4R$ is made from an iron plate of thickness $\frac{t}{4}$. Then the relation between the moment of inertia I_X and I_Y is

(a) $I_Y = 64 I_X$ (b) $I_Y = 32 I_X$ (c) $I_Y = 16 I_X$ (d) $I_Y = I_X$

Solution: (a) Moment of Inertia of disc $I = \frac{1}{2}MR^2 = \frac{1}{2}(\pi R^2 t\rho)R^2 = \frac{1}{2}\pi t\rho R^4$

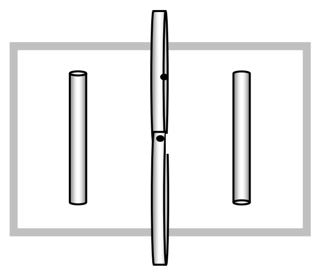
[As $M = V \times \rho = \pi R^2 t\rho$ where $t =$ thickness, $\rho =$ density]

$\therefore \quad \frac{I_y}{I_x} = \frac{t_y}{t_x}\left(\frac{R_y}{R_x}\right)^4$ [If ρ = constant]

$\Rightarrow \quad \frac{I_y}{I_x} = \frac{1}{4}(4)^4 = 64$ [Given $R_y = 4R_x$, $t_y = \frac{t_x}{4}$]

$\Rightarrow \quad I_y = 64 I_x$

Problem **07.** Four thin rods of same mass M and same length l, form a square as shown in figure. Moment of inertia of this system about an axis through centre O and perpendicular to its plane is



(a) $\frac{4}{3}Ml^2$

(b) $\frac{Ml^2}{3}$

(c) $\frac{Ml^2}{6}$

(d) $\frac{2}{3}Ml^2$

Solution: (a) Moment of inertia of rod AB about point $P = \frac{1}{12}Ml^2$

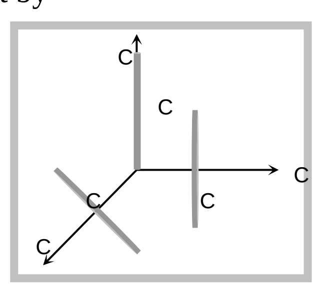
M.I. of rod AB about point O $= \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{1}{3}Ml^2$ [by the theorem of parallel axis]

and the system consists of 4 rods of similar type so by the symmetry $I_{System} = \frac{4}{3}Ml^2$.

Problem **08.** Three rods each of length L and mass M are placed along X, Y and Z-axes in such a way that one end of each of the rod is at the origin. The moment of inertia of this system about Z axis is

(a) $\frac{2ML^2}{3}$ (b) $\frac{4ML^2}{3}$ (c) $\frac{5ML^2}{3}$ (d) $\frac{ML^2}{3}$

Solution: (a) Moment of inertia of the system about z-axis can be find out by calculating the moment of inertia of individual rod about z-axis



$I_1 = I_2 = \frac{ML^2}{3}$ because *z*-axis is the edge of rod 1 and 2

and $I_3 = 0$ because rod in lying on *z*-axis

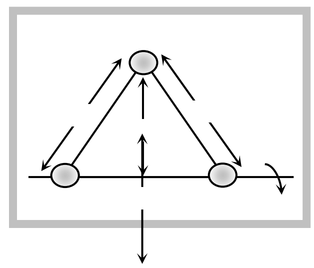
$\therefore\ I_{system} = I_1 + I_2 + I_3 = \frac{ML^2}{3} + \frac{ML^2}{3} + 0 = \frac{2ML^2}{3}$.

Problem **09.** Three point masses each of mass m are placed at the corners of an equilateral triangle of side *a*. Then the moment of inertia of this system about an axis passing along one side of the triangle is

(a) ma^2 (b) $3ma^2$ (c) $\frac{3}{4}ma^2$

(d) $\frac{2}{3}ma^2$

Solution: (c) The moment of inertia of system about *AB* side of triangle



$I = I_A + I_B + I_C$

$= 0 + 0 + mx^2$

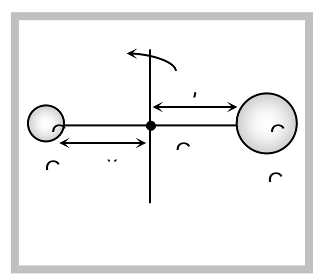
$= m\left(\frac{a\sqrt{3}}{2}\right)^2 = \frac{3}{4}ma^2$

Problem **10.** The moment of inertia of *HCl* molecule about an axis passing through its centre of mass and perpendicular to the line joining the H^+ and Cl^- ions will be, if the interatomic distance is 1 Å

(a) $0.61 \times 10^{-47}\ kg.m^2$ (b) $1.61 \times 10^{-47}\ kg.m^2$ (c) $0.061 \times 10^{-47}\ kg.m^2$

(d) 0

Solution: (b) If r_1 and r_2 are the respective distances of particles m_1 and m_2 from the centre of mass then



$m_1 r_1 = m_2 r_2 \Rightarrow 1 \times x = 35.5 \times (L - x) \Rightarrow x = 35.5\,(1 - x)$

$\Rightarrow x = 0.973\,Å$ and $L - x = 0.027\,Å$

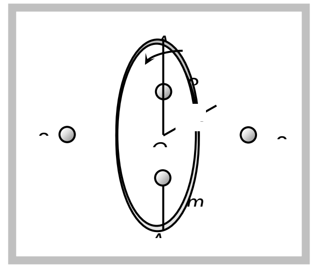
Moment of inertia of the system about centre of mass $I = m_1 x^2 + m_2 (L - x)^2$

$I = 1amu \times (0.973\,Å)^2 + 35.5\ amu \times (0.027\,Å)^2$

Substituting 1 *a.m.u.* = 1.67 × 10^{-27} *kg* and 1 Å = 10^{-10} *m*

$I = 1.62 \times 10^{-47}\,kg\,m^2$

Problem. **11.** Four masses are joined to a light circular frame as shown in the figure. The radius of gyration of this system about an axis passing through the centre of the circular frame and perpendicular to its plane would be



(a) $a/\sqrt{2}$

(b) $a/2$

(c) a

(d) $2a$

Solution: (c) Since the circular frame is massless so we will consider moment of inertia of four masses only.

$I = ma^2 + 2ma^2 + 3ma^2 + 2ma^2 = 8ma^2$ .....(i)

Now from the definition of radius of gyration $I = 8mk^2$ .....(ii)

comparing (i) and (ii) radius of gyration $k = a$.

***Problem* 12.** Two circular discs A and B are of equal masses and thickness but made of metals with densities d_A and d_B $(d_A > d_B)$. If their moments of inertia about an axis passing through centres and normal to the circular faces be I_A and I_B, then

(a) $I_A = I_B$ (b) $I_A > I_B$ (c) $I_A < I_B$ (d) $I_A >=< I_B$

Solution : (c) Moment of inertia of circular disc about an axis passing through centre and normal to the circular face

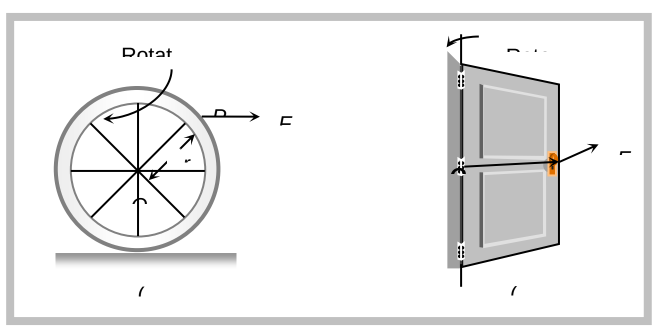
$I = \frac{1}{2}MR^2 = \frac{1}{2}M\left(\frac{M}{\pi t\rho}\right)$ [As $M = V\rho = \pi R^2 t\rho$ ∴ $R^2 = \frac{M}{\pi t \rho}$]

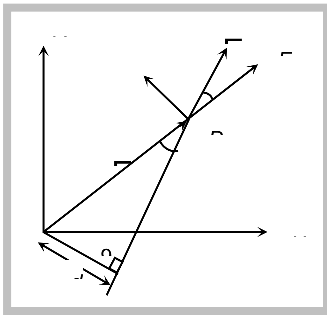
⇒ $I = \frac{M^2}{2\pi t\rho}$ or $I \propto \frac{1}{\rho}$ If mass and thickness are constant.

So, in the problem $\frac{I_A}{I_B} = \frac{d_B}{d_A}$ $\therefore I_A < I_B$ [As $d_A > d_B$]

7.14 Torque

If a pivoted, hinged or suspended body tends to rotate under the action of a force, it is said to be acted upon by a torque. or The turning effect of a force about the axis of rotation is called moment of force or torque due to the force.





If the particle rotating in xy plane about the origin under the effect of force $\vec{F}$ and at any instant the position vector of the particle is $\vec{r}$ then,

$$\text{Torque } \vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = rF\sin\phi$$

[where ϕ is the angle between the direction of $\vec{r}$ and $\vec{F}$]

(1) Torque is an axial vector. *i.e.*, its direction is always perpendicular to the plane containing vector $\vec{r}$ and $\vec{F}$ in accordance with right hand screw rule. For a given figure the sense of rotation is anti-clockwise so the direction of torque is perpendicular to the plane, outward through the axis of rotation.

(2) Rectangular components of force

$$\vec{F}_r = F\cos\phi = \text{radial component of force}$$

$$\vec{F}_\phi = F\sin\phi = \text{transverse component of force}$$

As $\tau = rF\sin\phi$

or $\tau = rF_\phi$ = (position vector) □ (transverse component of force)

Thus the magnitude of torque is given by the product of transverse component of force and its perpendicular distance from the axis of rotation *i.e.*, Torque is due to transverse component of force only.

(3) As $\tau = rF\sin\phi$

or □ $\tau = F(r\sin\phi) = Fd$ [As $d = r\sin\phi$ from the figure]

i.e. Torque = Force x Perpendicular distance of line of action of force from the axis of rotation.

Torque is also called as moment of force and *d* is called moment or lever arm.

(4) Maximum and minimum torque : As $\vec{\tau} = \vec{r} \times \vec{F}$ or $\tau = rF\sin\phi$

(5) For a given force and angle, magnitude of torque depends on *r*. The more is the value of *r*, the more will be the torque and easier to rotate the body.

Example : (i) Handles are provided near the free edge of the Planck of the door.

(ii) The handle of screw driver is taken thick.

(iii) In villages handle of flourmill is placed near the circumference.

(iv) The handle of hand-pump is kept long.

(v) The arm of wrench used for opening the tap, is kept long.

(6) Unit : *Newton-metre* (M.K.S.) and *Dyne-cm* (C.G.S.)

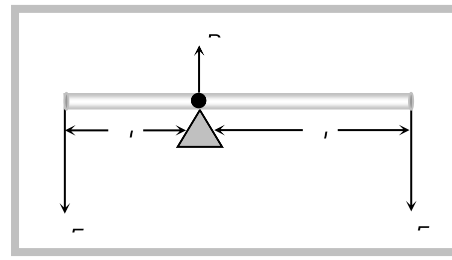
(7) Dimension : $[ML^2T^{-2}]$.

(8) If a body is acted upon by more than one force, the total torque is the vector sum of each torque.

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \ldots\ldots$$

(9) A body is said to be in rotational equilibrium if resultant torque acting on it is zero *i.e.* $\Sigma \vec{\tau} = 0$.

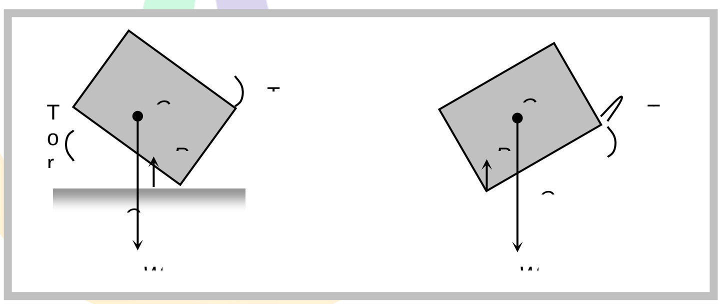
(10) In case of beam balance or see-saw the system will be in rotational equilibrium if,



$\vec{\tau}_1 + \vec{\tau}_2 = 0$ or $F_1 l_1 - F_2 l_2 = 0 \therefore F_1 l_1 = F_2 l_2$

However if, $\vec{\tau}_1 > \vec{\tau}_2$, L.H.S. will move downwards and if $\vec{\tau}_1 < \vec{\tau}_2$. R.H.S. will move downward. and the system will not be in rotational equilibrium.

(11) On tilting, a body will restore its initial position due to torque of weight about the point *O* till the line of action of weight passes through its base on tilting, a body will topple due to torque of weight about *O*, if the line of action of weight does not pass through the base.



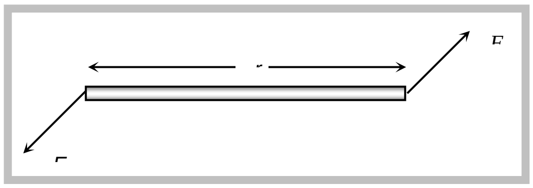
(12) Torque is the cause of rotatory motion and in rotational motion it plays same role as force plays in translatory motion *i.e.*, torque is rotational analogue of force. This all is evident from the following correspondences between rotatory and translatory motion.

dt	dt

7.15 Couple

A special combination of forces even when the entire body is free to move can rotate it. This combination of forces is called a couple.

(1) A couple is defined as combination of two equal but oppositely directed force not acting along the same line. The effect of couple is known by its moment of couple or torque by a couple $\vec{\tau} = \vec{r} \times \vec{F}$.

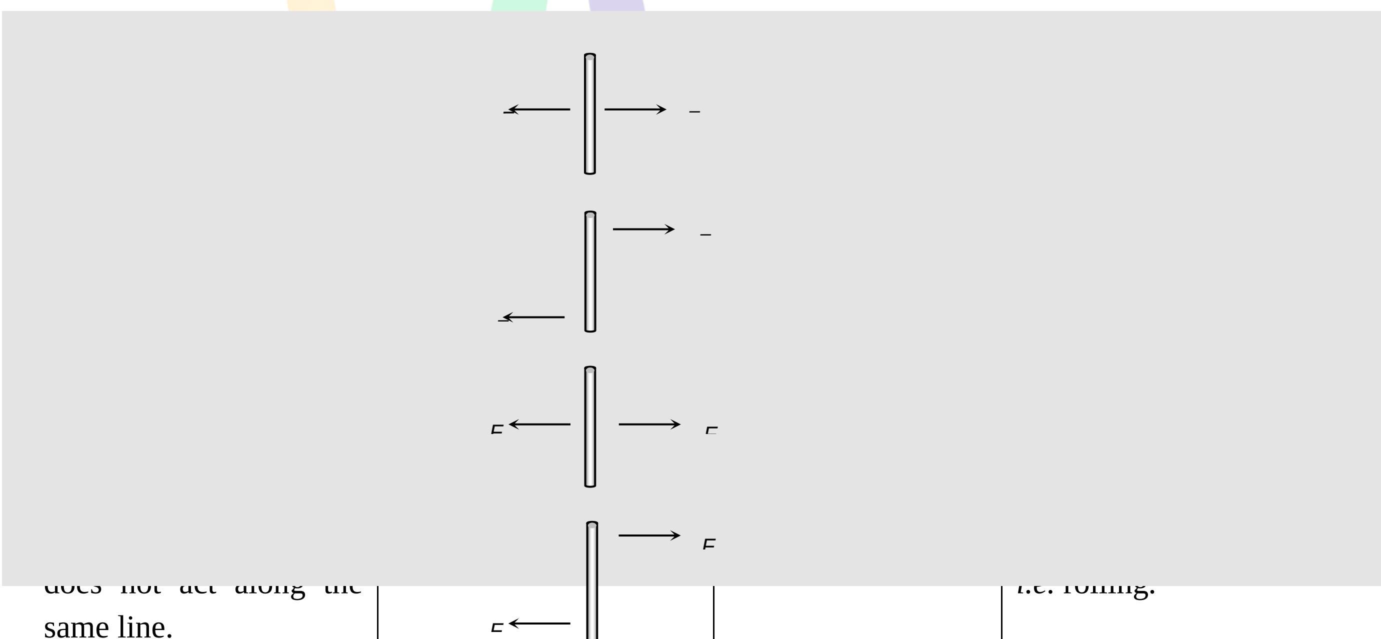


(2) Generally both couple and torque carry equal meaning. The basic difference between torque and couple is the fact that in case of couple both the forces are externally applied while in case of torque one force is externally applied and the other is reactionary.

(3) Work done by torque in twisting the wire $W = \frac{1}{2} C\theta^2$.

Where $\tau = C\theta$; C is known as twisting coefficient or couple per unit twist.

7.16 Translatory and Rotatory Equilibrium



same line.			i.e. rolling.

Problems based on torque and couple

Problem **14.** A force of $(2\hat{i} - 4\hat{j} + 2\hat{k})N$ acts at a point $(3\hat{i} + 2\hat{j} - 4\hat{k})$ *metre* from the origin. The magnitude of torque is

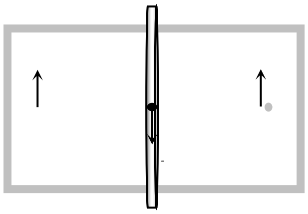
(a) Zero (b) 24.4 *N-m* (c) 0.244 *N-m* (d) 2.444 *N-m*

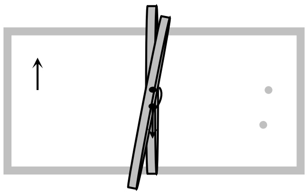
Solution: (b) $\vec{F} = (2\hat{i} - 4\hat{j} + 2\hat{k})\,N$ and $\vec{r} = (3i + 2 - 4\hat{k})$ *meter*

Torque $\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -4 \\ 2 & -4 & 2 \end{vmatrix} \Rightarrow \vec{\tau} = -12\hat{i} - 14\hat{j} - 16\hat{k}$ and $|\vec{\tau}| = \sqrt{(-12)^2 + (-14)^2 + (-16)^2} =$ 24.4 *N-m*

Problem **15.** A horizontal heavy uniform bar of weight *W* is supported at its ends by two men. At the instant, one of the men lets go off his end of the rod, the other feels the force on his hand changed to

(a) W (b) $\frac{W}{2}$ (c) $\frac{3W}{4}$ (d) $\frac{W}{4}$





Solution: (d) Let the mass of the rod is *M* $\therefore$ Weight (*W*) = *Mg*

Initially for the equilibrium $F + F = Mg$ $\Rightarrow F = Mg/2$

When one man withdraws, the torque on the rod

$\tau = I\alpha = Mg\frac{l}{2}$

$\Rightarrow \frac{Ml^2}{3}\alpha = Mg\frac{l}{2}$ [As $I = Ml^2/3$]

$\Rightarrow$Angular acceleration $\alpha = \frac{3}{2}\frac{g}{l}$

and linear acceleration $a = \frac{l}{2}\alpha = \frac{3g}{4}$

Now if the new normal force at *A* is F' then $Mg - F' = Ma$

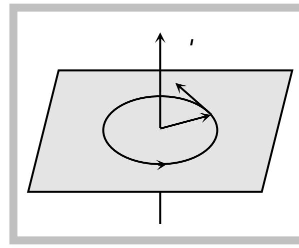
$\Rightarrow F' = Mg - Ma = Mg - \frac{3Mg}{4} = \frac{Mg}{4} = \frac{W}{4}$.

7.17 Angular Momentum

The turning momentum of particle about the axis of rotation is called the angular momentum of the particle.

or

The moment of linear momentum of a body with respect to any axis of rotation is known as angular momentum. If $\vec{P}$ is the linear momentum of particle and $\vec{r}$ its position vector from the point of rotation then angular momentum.



$$\vec{L} = \vec{r} \times \vec{P}$$

$$\vec{L} = r\,P\sin\phi\,\hat{n}$$

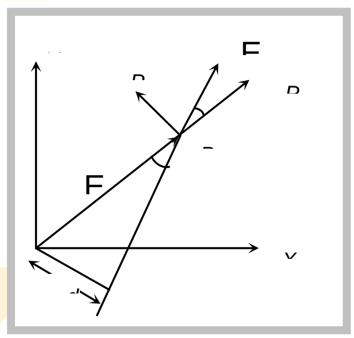
Angular momentum is an axial vector *i.e.* always directed perpendicular to the plane of rotation and along the axis of rotation.

(1) S.I. Unit : $kg\text{-}m^2\text{-}s^{-1}$ or *J-sec*.

(2) Dimension : $[ML^2T^{-1}]$ and it is similar to Planck's constant (h).

(3) In cartesian co-ordinates if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$

Then $\vec{L} = \vec{r} \times \vec{P} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix} = (yP_z - zP_y)\hat{i} - (xP_z - zP_x)\hat{j} + (xP_y - yP_x)\hat{k}$



(4) As it is clear from the figure radial component of momentum $\vec{P}_r = P\cos\phi$

Transverse component of momentum $\vec{P}_\phi = P\sin\phi$

So magnitude of angular momentum $L = r\,P\sin\phi$

$$L = r\,P_\phi$$

□ Angular momentum = Position vector × Transverse component of angular momentum

i.e., The radial component of linear momentum has no role to play in angular momentum.

(5) Magnitude of angular momentum $L = P\ (r\sin\phi) = L = Pd$ [As $d = r\sin\phi$ from the figure.]

∴ Angular momentum = (Linear momentum) ×□(Perpendicular distance of line of action of force from the axis of rotation)

(6) Maximum and minimum angular momentum : We know $\vec{L} = \vec{r} \times \vec{P}$

∴ $\vec{L} = m[\vec{r} \times \vec{v}] = m\,v\,r\sin\phi = P\,r\sin\phi$ [As $\vec{P} = m\,\vec{v}$]

$L_{maximum} = mvr$	When $\lvert\sin\phi\rvert = \max = 1$ *i.e.*, $\phi = 90°$	$\vec{v}$ is perpendicular to $\vec{r}$

$L_{\text{minimum}} = 0$	When $\lvert \sin\phi \rvert = \min = 0$ *i.e.* $\phi = 0°$ or $180°$	$\vec{v}$ is parallel or anti-parallel $\vec{r}$

(7) A particle in translatory motion always have an angular momentum unless it is a point on the line of motion because $L = mvr\sin\phi$ and $L > 1$ if $\phi \neq 0^o$ or 180^o

(8) In case of circular motion, $\vec{L} = \vec{r} \times \vec{P} = m(\vec{r} \times \vec{v}) = mvr\sin\phi$

$\therefore$ $L = mvr = mr^2\omega$ [As $\vec{r} \perp \vec{v}$ and $v = r\omega$]

or $L = I\omega$ [As $mr^2 = I$]

In vector form $\vec{L} = I\vec{\omega}$

(9) From $\vec{L} = I\vec{\omega}$ $\therefore$ $\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha} = \vec{\tau}$ [As $\frac{d\vec{\omega}}{dt} = \vec{\alpha}$ and $\vec{\tau} = I\vec{\alpha}$]

i.e. the rate of change of angular momentum is equal to the net torque acting on the particle. [Rotational analogue of Newton's second law]

(10) If a large torque acts on a particle for a small time then 'angular impulse' of torque is given by

$$\vec{J} = \int \vec{\tau}\, dt = \vec{\tau}_{av} \int_{t_1}^{t_2} dt$$

or Angular impulse $\vec{J} = \vec{\tau}_{av}\,\Delta t = \Delta\vec{L}$

$\therefore$ Angular impulse = Change in angular momentum

(11) The angular momentum of a system of particles is equal to the vector sum of angular momentum of each particle *i.e.*, $\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + ... + \vec{L}_n$.

(12) According to Bohr theory angular momentum of an electron in n^{th} orbit of atom can be taken as,

$$L = n\frac{h}{2\pi}$$ [where n is an integer used for number of orbit]

7.18 Law of Conservation of Angular Momentum

Newton's second law for rotational motion $\vec{\tau} = \frac{d\vec{L}}{dt}$

So if the net external torque on a particle (or system) is zero then $\frac{d\vec{L}}{dt} = 0$

i.e. $\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + ...$ = constant.

Angular momentum of a system (may be particle or body) remains constant if resultant torque acting on it zero.

As $L = I\omega$ so if $\vec{\tau} = 0$ then $I\omega = \text{constant}$ $\therefore$ $I \propto \frac{1}{\omega}$

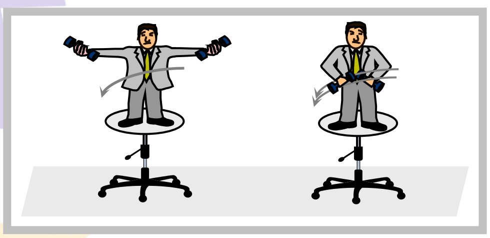
Since angular momentum $I\omega$ remains constant so when I decreases, angular velocity ω increases and vice-versa.

Examples of law of conservation of angular momentum :

(1) The angular velocity of revolution of a planet around the sun in an elliptical orbit increases when the planet come closer to the sun and vice-versa because when planet comes closer to the sun, it's moment of inertia I decreases there fore ω increases.

(2) A circus acrobat performs feats involving spin by bringing his arms and legs closer to his body or vice-versa. On bringing the arms and legs closer to body, his moment of inertia I decreases. Hence ω increases.

(3) A person-carrying heavy weight in his hands and standing on a rotating platform can change the speed of platform. When the person suddenly folds his arms. Its moment of inertia decreases and in accordance the angular speed increases.



(4) A diver performs somersaults by Jumping from a high diving board keeping his legs and arms out stretched first and then curling his body.

(5) Effect of change in radius of earth on its time period

Angular momentum of the earth $L = I\omega = \text{constant}$

$$L = \frac{2}{5}MR^2 \times \frac{2\pi}{T} = \text{constant}$$

$\therefore$ $T \propto R^2$

[if M remains constant]

If R becomes half then time period will become one-fourth *i.e.* $\frac{24}{4} = 6hrs.$

Problems based on angular momentum

NCRTT .16 Three bodies, a ring, a solid cylinder and a solid sphere roll down the same inclined plane without slipping. They start from rest. The radii of the bodies are identical. Which of the bodies reaches the ground with maximum velocity?

***NCRTT* 7.15** A cord of negligible mass is wound round the rim of a fly wheel of mass 20 kg and radius 20 cm. A steady pull of 25 N is applied on the cord as shown in Fig. 7.35. The flywheel is mounted on a horizontal axle with frictionless bearings.

(a) Compute the angular acceleration of the wheel.

(b) Find the work done by the pull, when 2m of the cord is unwound.

(c) Find also the kinetic energy of the wheel at this point. Assume that the wheel starts from rest.

(d) Compare answers to parts (b) and (c).

NCRTT* 7.15 *7.8 A non-uniform bar of weight *W* is suspended at rest by two strings of Negligible weight as shown in Fig.7.39. The angles made by the strings with the vertical are 36.9° and 53.1° respectively. The bar is 2 m long. Calculate the distance *d* of the centre of gravity of the bar from its left end.

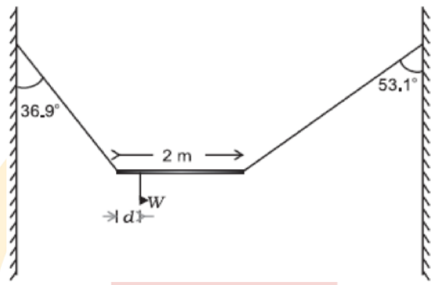


Fig. 7.39.

NCRTT* 7.15 *7.9 A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.

***<u>Problem</u>* 15.** A thin circular ring of mass *M* and radius *R* is rotating about its axis with a constant angular velocity ω. Four objects each of mass *m*, are kept gently to the opposite ends of two perpendicular diameters of the ring. The angular velocity of the ring will be

(a) $\frac{M\omega}{M+4m}$ (b) $\frac{(M+4m)\omega}{M}$ (c) $\frac{(M-4m)\omega}{M+4m}$ (d) $\frac{M\omega}{4m}$

Solution: (a) Initial angular momentum of ring $= I\omega = MR^2\omega$

If four object each of mass *m*, and kept gently to the opposite ends of two perpendicular diameters of the ring then final angular momentum = $(MR^2 + 4mR^2)\omega'$

By the conservation of angular momentum

Initial angular momentum = Final angular momentum

$MR^2\omega = (MR^2 + 4mR^2)\omega' \Rightarrow \omega' = \left(\frac{M}{M+4m}\right)\omega$.

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