RELATIONS AND FUNCTIONS

1. RELATIONS

DEFINITION

A relation *R*, from a non-empty set *A* to another non-empty set *B*, is a subset of $A \times B$

Equivalently, any subset of $A \times B$ is relation from *A* to *B*.

Thus, *R* is a relation from *A* to *B* ⇔ $R \subseteq A \times B$

$$\Leftrightarrow \quad R \subseteq \{(a,\ b) : a \in A,\ \ b \in B\}$$

Example: Let $A = \{1,\ 2\},\ B = \{a,\ b,\ c\}$

Let $R = \{(1,\ a),\ (1,\ c)\}$

Here *R* is a subset of $A \times B$ and hence it is a relation from *A* to *B*.

2. DOMAIN AND RANGE OF A RELATION

1. DOMAIN OF A RELATION

Let *R* be a relation from *A* to *B*. The domain of relation *R* is the set of all those elements $a \in A$ such that $(a,\ b) \in R$ for some $b \in B$.

Thus Dom (*R*) = $\{a \in A : (a,\ b) \in R \text{ for some } b \in B\}$

Thus domain of *R* = set of first components of all the ordered pair which belong to *R*.

2. RANGE OF A RELATION

Let R be a relation from A to B. The range of R is the set of all those elements $b \in R$ such that $(a, b) \in R$ for some $a \in A$.

Thus range of $R = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$.

Range of R = set of second components of all the ordered pairs which belong to R.

Set B is called as codomain of relation R.

Example: Let $A = \{2, 3, 5\}$ and $B = \{4, 7, 10, 8\}$

Let $aRb \Leftrightarrow a$ divides b

Then Dom $R = \{2, 5\}$ and range of $R = \{4, 10, 8\}$

Codomain of $R = B = \{4, 7, 10, 8\}$

Example: Let $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8\}$

Let R be a relation defined from A to B by $xRy \Leftrightarrow y$ is double of $x, \forall\ x \in A$

Then $1R2, 2R4, 3R6$

$\therefore\ R = \{(1, 2), (2, 4), (3, 6)\}$

3. TYPES OF RELATIONS ON A SET

1. Empty relation: A relation R on a set A is said to be an empty relation or a void relation if $R = \varphi$.

Example: Let $A = \{1, 2, 3, 4, 5\}$

Let $R = \{(a, b) : a - b = 12 \text{ and } a, b \in A\}$.

We observe that $a - b \neq 12$ for any two elements of A. Therefore,

$(a, b) \notin R$ for any $a, b \in A$.

$\Rightarrow$ R does not contain any element of $A \times A$

$\Rightarrow$ R is empty set

$\Rightarrow$ R is the empty relation (void relation) on A

RELATIONS AND FUNCTIONS

2. Universal relation: A relation R on a set A is said to be universal relation on A if $R = A \times A$.

Example: Let $A = \{1, 2\}$.

Let $R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

Here $R = A \times A$

Hence R is the universal relation on A.

3. Identity relation: A relation R on a set A is said to be identity relation on A if $R = \{(a, b) : a \in A, b \in A \text{ and } a = b\}$.

Thus identity relation $R = \{(a, a) : \forall\, a \in A\}$.

Identity relation on set A is also denoted by I_A.

Example: Let $A = \{1, 2, 3, 4\}$

Then $I_A = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

Note: (i) Empty relation and universal relation are called **trivial relations.**

(ii) In an identity relation on A, every element of A should be related to itself only.

Example: **Let $A = \{3, 5\}$, $B = \{7, 11\}$**

Let $R = \{(a, b) : a \in A, b \in B, a - b \text{ is even}\}$

Show that R is an universal relation from A to B.

Solution: Given, $A = \{3, 5\}, B = \{7, 11\}$

Now, $R = \{(a, b) : a \in A, b \in B \text{ and } a - b \text{ is even}\}$

$= \{(3, 7), (3, 11), (5, 7), (5, 11)\}$

Also $A \times B = \{(3, 7), (3, 11), (5, 7), (5, 11)\}$

Clearly, $R = A \times B$

Hence R is an universal relation from A to B.

4. REFLEXIVE, SYMMETRIC AND TRANSITIVE RELATIONS

1. Reflexive relation: A relation R on a set A is said to be reflexive if $(a, a) \in R, \ \forall\, a \in A$.

RELATIONS AND FUNCTIONS

Example: Let $A = \{1, 2, 3\}$.

Let $R_1 = \{(1, 1), (2, 2), (3, 3)\}$

$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3)\}$

$R_3 = \{(2, 2), (2, 3), (3, 2), (1, 1)\}$

Here R_1 and R_2 are reflexive relations on *A*.

R_3 is not a reflexive relation on *A* as (3, 3) $\notin A$.

Note: The identity relation is always a reflexive relation but a reflexive relation may or may not be the identity relation. In the examples given above R_1 is both reflexive and identity relation on *A* whereas R_2 is a reflexive relation on *A* but not an identity relation on *A*.

2. **Symmetric relation:** A relation *R* on a set *A* is said to be a symmetric relation if $(a, b) \in R \Rightarrow (b, a) \in R$, where $a, b \in A$.

Example: Let $A = \{1, 2, 3\}$.

Let $R_1 = \{(1, 2), (2, 1)\}$

$R_2 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$

Here R_1 and R_2 are symmetric relations on *A*.

Let $R_3 = \{(2, 3), (3, 2), (2, 2)\}$ and $R_4 = \{(2, 3), (3, 1), (1, 3)\}$.

Then R_3 is a symmetric relation on *A* because

$(2, 3) \in R_3 \Rightarrow (3, 2) \in R_3$

But R_4 is not a symmetric relation on *A* because

$(2, 3) \in R_4$ and $(3, 2) \notin R_4$

3. **Transitive relation:** A relation *R* on a set *A* is said to be a transitive relation if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$, where $a, b, c \in A$.

Example: Let $A = \{1, 2, 3\}$

Let R_1 = {1, 2), (2, 3), (1, 3), (3, 2)}.

Let R_2 = {(1, 3), (3, 2), (1, 2)}.

Then R_1 is not transitive relation on A because (2, 3) $\in R$ and (3, 2) $\in R$ but (2, 2) $\notin R$.

Finally R_2 is also a transitive relation.

Thus R is antisymmetric if $a \neq b$, then both (a, b) and (b, a) cannot belong to R. One of them may belong to R.

Example: Let A = {1, 2, 3}.

Let $R_1 = \{(1, 2), (1, 3), (1, 1)\}$

$R_2 = \{(1, 2)\}$

$R_3 = \{(1, 2), (2, 1)\}$

Here R_1 and R_2 are antisymmetric relations on A but R_3 is not antisymmetric relation on A.

Note: A relation which is not symmetric is not necessarily antisymmetric.

Example: Let A = {1, 2, 3}.

Let R = {(1, 2), (2, 1), (2, 3)}.

Here R is not symmetric as (2, 3) $\in R$ but (3, 2) $\notin R$

Also R is not antisymmetric because (1, 2) $\in R$ and (2, 1) $\in R$ but $1 \neq 2$

5. EQUIVALENCE RELATION ON A SET

In this section we shall define a very interesting relation on a set which plays a very significant role in mathematics. This relation has been given the name equivalence relation.

DEFINITION

Let A be a non-empty set, then a relation R on A is said to be an equivalence relation if

(i) R is reflexive i.e., $(a, a) \in R, \forall\ a \in A$.

(ii) R is symmetric i.e., $(a, b) \in R \Rightarrow (b, a) \in R$, where $a, b \in A$.

(iii) R is transitive i.e., $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow (a, c) \in R$, where $a, b, c \in A$.

Example: Let $A = \{1, 2, 3\}$.

Let a relation R be defined on A as

$R = \{(1, 2), (1, 1), (2, 1), (2, 2), (3, 3)\}$

Then R is reflexive, symmetric and transitive. So R is an equivalence relation on A.

6. DEFINITION OF FUNCTION

Function is the most important part of calculus as it deals with dependency of one physical quantity on the other.

Let X and Y be any two non-empty sets. "*A* function from X to Y is *a* rule or correspondence that assigns to each element of set X, one and only one element of set Y".

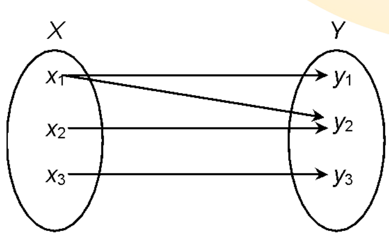
$f : X \to Y$ where $y = f(x)$, $x \in X$ and $y \in Y$.

'y' is the image of 'x' under 'f'

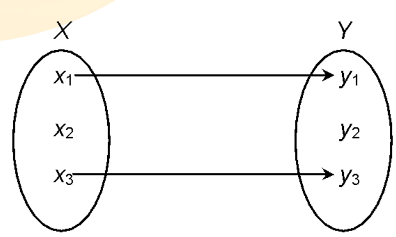
(i) *A* mapping $f : X \to Y$ is said to be *a* function if each element in the set X has it's image in set Y.

(ii) Every element in set X should have one and only one image.

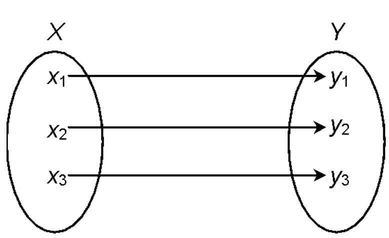
For $y = f(x)$ (y is *a* function of x), x is called independent variable and y is called dependent variable.



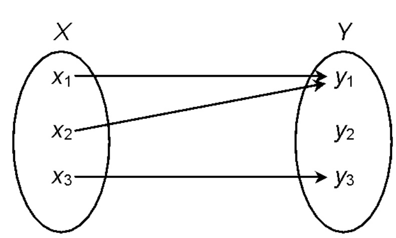
Not a function



Not a function



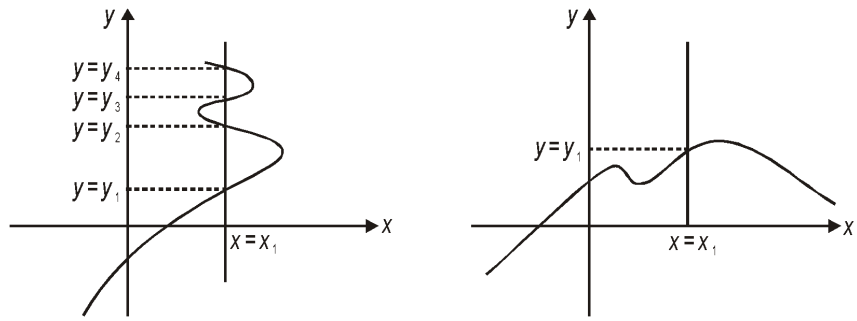
Function



Function

RELATIONS AND FUNCTIONS

If *a* line parallel to *y*-axis cuts the graph of the relation between *x* and *y*, in not more then one point, the relation is said to be a function in *x* or in other words, *y* is said to be a function of *x*.



y* is not a function of *x

y* is a function of *x

Example: **For the following *y* can be a function of *x*. ($x \in R$, $y \in R$):**

$\boldsymbol{y^2 = 4ax}$

***Solution*:** $y^2 = 4ax \Rightarrow y = \pm\sqrt{4ax}$

This is not a function.

Example: **If $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$, show that $f(x)+f(y) = f\left(\dfrac{x+y}{1+xy}\right)$.**

***Solution*:** We have, $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$

$$\therefore \quad f(x)+f(y) = \log\left(\frac{1+x}{1-x}\right) + \log\left(\frac{1+y}{1-y}\right) = \log\left[\left(\frac{1+x}{1-x}\right)\left(\frac{1+y}{1-y}\right)\right]$$

$$= \log\left(\frac{1+x+y+xy}{1-x-y+xy}\right) \qquad \ldots(i)$$

and

$$f\left(\frac{x+y}{1+xy}\right) = \log\left(\frac{1+\dfrac{x+y}{1+xy}}{1-\dfrac{x+y}{1+xy}}\right) = \log\left(\frac{1+xy+x+y}{1+xy-x-y}\right) \qquad \ldots(ii)$$

From (i) and (ii), we obtain

$$f(x)+f(y) = f\left(\frac{x+y}{1+xy}\right)$$

7. DOMAIN AND CODOMAIN OF FUNCTION

Let $f : X \rightarrow Y$ be *a* function.

Set 'X' is called domain of the function 'f'.

Set 'Y' is called the co-domain of the function 'f',

Example: **Find the domain of the following functions:**

$$f(x) = \sqrt{1-2x}$$

Solution: $1 - 2x \geq 0$

Hence, domain (f) = $\left(-\infty, \frac{1}{2}\right]$

8. RANGE OF FUNCTION

Set of images of all elements of set X is called the range of the function. range could be a subset of co-domain

Example: **Find the range of the following functions:**

$f(x) = 2\sin x + 3\cos x$

Solution: $f(x) = 2\sin x + 3\cos x$

$$= \sqrt{13}\left(\sin x.\frac{2}{\sqrt{13}} + \cos x.\frac{3}{\sqrt{13}}\right) = \sqrt{13}\sin(x+\theta)$$, where $\tan\theta = \frac{3}{2}$

$\Rightarrow$ Range (f) = $[-\sqrt{13}, \sqrt{13}]$

9. DIFFERENT TYPES OF FUNCTIONS

Algebraic functions: Functions consisting of finite number of terms involving powers and roots of the independent variable with the operations +, −, ×, ÷ are called algebraic functions.

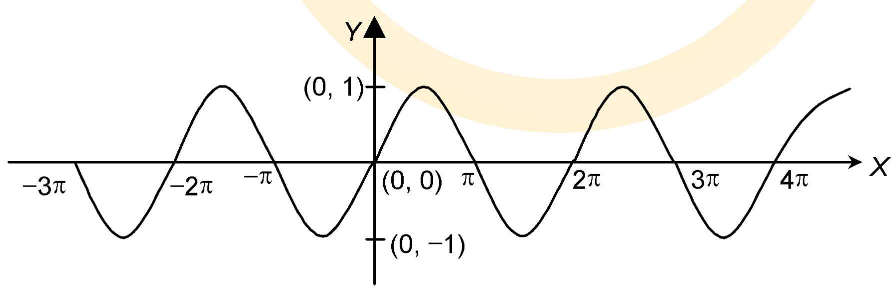
Polynomial functions: $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx^n$,

where $a_0, a_1, a_2, \ldots\ldots, a_n \in \mathbf{R}$; $a_n \neq 0$ is said to be a polynomial function of degree n.

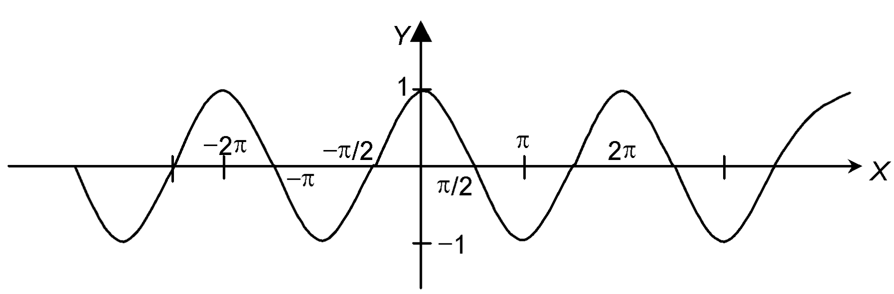
Trigonometric/circular functions: Functions involving trigonometric ratios are called trigonometric functions.

Examples:

(a) $y = f(x) = \sin x$

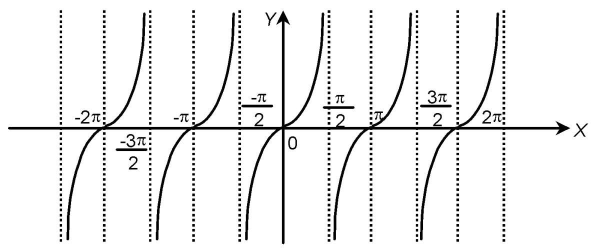


(b) $y = f(x) = \cos x$

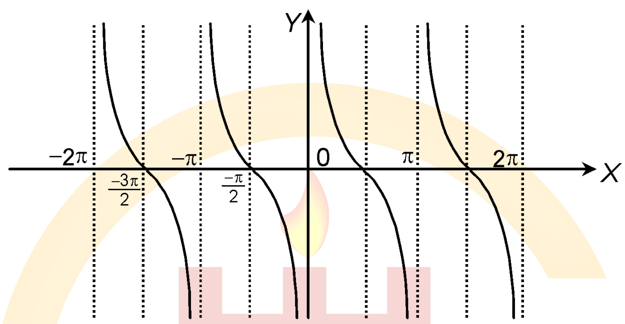


RELATIONS AND FUNCTIONS

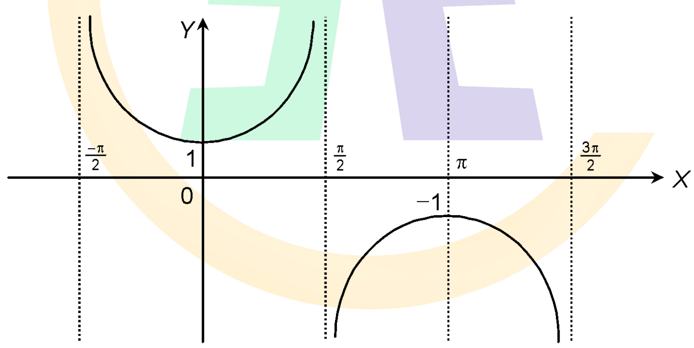
(c) $y = f(x) = \tan x$



(d) $y = f(x) = \cot x$

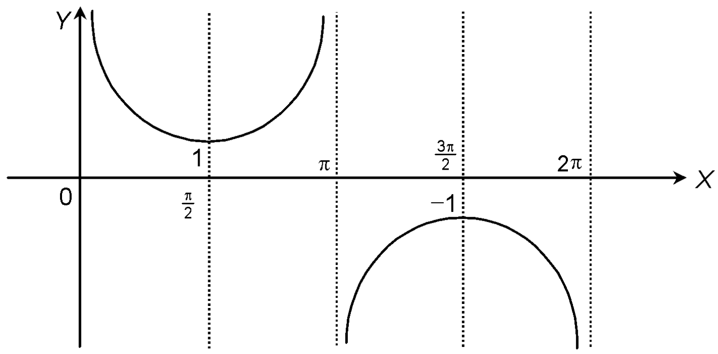


(e) $y = f(x) = \sec x$



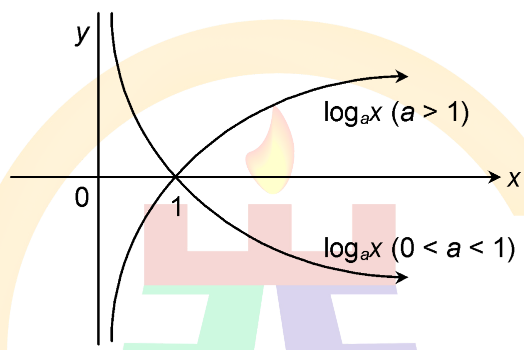
(f) $y = f(x) = \text{cosec}x$

RELATIONS AND FUNCTIONS

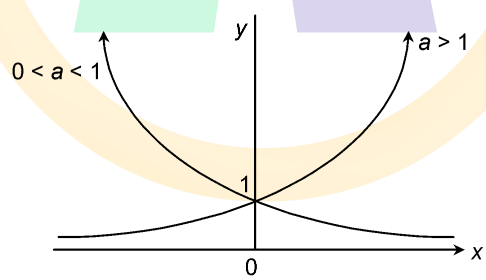


Logarithmic function: If $a > 0$, $a \neq 1$, then the function $y = \log_a x$, $x \in R^+$ is called a logarithmic function

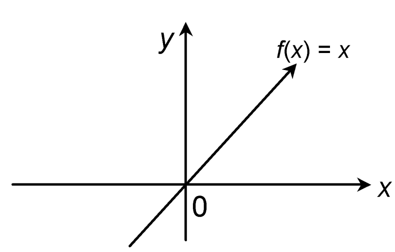
Logarithmic function is the inverse of the exponential function.



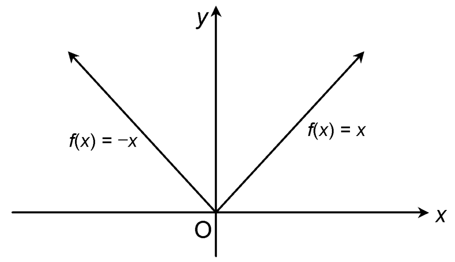
Exponential function: If $a > 0$, $a \neq 1$, then the function defined by $y = a^x$, $x \in R$ is called an exponential function.



Identity function: $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x$.



Absolute value function: $f : \mathbf{R} \to \mathbf{R}$, $f(x) = |x|$.



Hence $|x| = \sqrt{x^2}$

RELATIONS AND FUNCTIONS

Basic properties of | *x* |

- Geometrical meaning of $|x - y|$ is the distance between x and y.
- $| x + y | \leq | x | + | y |$

Example: **Solve the inequalities for real values of *x*: $|x - 3| > 5$.**

***Solution*:** $|x - 3| > 5$

$\Rightarrow \quad x - 3 < -5$ or $x - 3 > 5$

$\Rightarrow \quad x < -2$ or $x > 8 \Rightarrow x \in (-\infty, -2) \cup (8, \infty)$

10. COMPOSITE FUNCTION

If range $(f) \subset$ domain (g), we define *gof* by the rule

$(gof)(x) = g(f(x))$ for all $x \in D_1$.

if range $(g) \subset$ domain (f), we define *fog* by the rule

$(fog)(x) = f(g(x))$ for all $x \in D_2$.

$h(x) = gof(x)$

To obtain $h(x)$, we first take the f-image of an element $x \in D_1$ so that $f(x) \in D_2$, which is the domain of $g(x)$. Then take g-image of $f(x)$, i.e. $g(f(x))$ which would be an element of D. The function h defined above is called the composition of f and g.

Domain $(gof) = \{x : x \in \text{domain } (f), f(x) \in \text{domain } (g)\}$

Example: **Let *f* be the sine function and let *g* be the function 2*x*. Find**

(i) *fog* (ii) *gof* (iii) *fof* (iv) *gog*

***Solution*:** We have $f(x) = \sin x$ and $g(x) = 2x$

(i) $(fog)(x) = f(g(x)) = f(2x) = \sin 2x$

(ii) $(gof)(x) = g(f(x)) = g(\sin x) = 2\sin x$

(iii) $(fof)(x) = f(f(x)) = f(\sin x) = \sin(\sin x)$

(iv) $(gog)(x) = g(g(x)) = g(2x) = 2(2x) = 4x$

11. CLASSIFICATION OF FUNCTIONS

1. ONE-ONE AND MANY-ONE FUNCTIONS

If each element in the domain of a function has a distinct image in the co-domain, the function is said to be one-one. One-one functions are also called injective function.

If there are at least two elements in the domain whose images are the same, the function is known as many-one.

RELATIONS AND FUNCTIONS

Lines drawn parallel to the *x*-axis from the each corresponding image point shall intersect the graph of $y = f(x)$ at one (and only one) point if $f(x)$ is one-one and there will be at least one line parallel to *x*-axis that will cut the graph at least at two different points, if $f(x)$ is many-one and vice versa.

A many-one function can be made one-one by redefining the domain of the original function.

Methods to determine one-one and many-one:

- Let x_1 , $x_2 \in$ domain of f and if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ for every x_1, x_2 in the domain, then f is one-one else many-one.
- Conversely if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for every x_1, x_2 in the domain, then f is one-one else many-one.
- If the function is entirely increasing or decreasing in the domain, then f is one-one else many-one.
- Any continuous function $f(x)$, which has at least one local maxima or local minima is many-one.
- All even functions are many one.
- All polynomials of even degree defined in *R* have at least one local maxima or minima and hence are many one in the domain *R*.

Example: **Show that the function** $\boldsymbol{f(x)=\dfrac{x^2-8x+18}{x^2+4x+30}}$ **is not one-one.**

***Solution*:** A function is one-one if $f(x_1)=f(x_2)\Rightarrow x_1=x_2$ (only)

Now $f(x_1)=f(x_2)\Rightarrow \dfrac{x_1^2-8x_1+18}{x_1^2+4x_1+30}=\dfrac{x_2^2-8x_2+18}{x_2^2+4x_2+30}$

$$\Rightarrow 12x_1^2x_2-12x_1x_2^2+12x_1^2-12x_2^2-312x_1+312x_2=0$$

$$\Rightarrow (x_1-x_2)\{12x_1x_2+12(x_1+x_2)-312\}=0$$

$$\Rightarrow x_1=x_2 \text{ or } x_1=\frac{26-x_2}{1+x_2}$$

Since $f(x_1)=f(x_2)$ does not imply $x_1=x_2$ alone, $f(x)$ is not a one-one function.

---

2. ONTO AND INTO FUNCTIONS

If each element in the codomain '*Y*' has at least one pre-image in the domain '*X*' that is, for every $y \in Y$ there exists at least one element $x \in X$ such that $f(x) = y$, then f is onto.

If there exists at least one corresponding element in the codomain '*Y*' which is not an image of any element in the domain *X*, then f is into.

Onto function is also called surjective function.

RELATIONS AND FUNCTIONS

Methods to determine whether a function is onto or into :

· If range = codomain, then *f* is onto. If range is a proper subset of codomain, then *f* is into.

Any polynomial function $f : \mathbf{R} \to \mathbf{R}$ is onto, if degree is odd; into, if degree of *f* is even.

Example: **Let $f : N \to I$ be a function defined as $f(x) = x - 1000$. Show that *f* is an into function.**

***Solution*:** Let $f(x) = y = x - 1000$

$\Rightarrow \quad x = y + 1000 = g(y)$ (say)

Here $g(y)$ is defined for each $y \in \mathbf{I}$, but $g(y) \notin N$ for $y \le -1000$.

Hence *f* is into.

---

3. BIJECTIVE FUNCTION

The function which is one-one and onto both is called bijective function.

Inverse of a function

If $f : X \to Y$ be a function defined by $y = f(x)$ such that *f* is both one-one and onto, then there exists a unique function $g : Y \to X$ such that for each $y \in Y$, $g(y) = x$ if and only if $y = f(x)$. The function *g* so defined is called the inverse of *f* and denoted by f^{-1}. Also if *g* is the inverse of *f*, then *f* is the inverse of *g* and the two functions *f* and *g* are said to be inverses of each other.

The range of the original function becomes the domain of the inverse function and domain of the original function becomes the range of the inverse function.

$fof^{-1}(x) = f[f^{-1}(x)] = x$ always, it means *f* and f^{-1} are symmetric about the line $y = x$.

4. EVEN AND ODD FUNCTION

Even function: *f* is said to be an even function of *x*, if $f(x) = f(-x)$ for all $x, -x \in$ domain (*f*).

Odd function: *f* is said to be an odd function of *x*, if $f(-x) = -f(x)$ for all $x, -x \in$ domain (*f*).

Note:

(i) To prove a function even, prove $f(x) - f(-x) = 0$

(ii) To prove a function $f(x)$ odd, prove $f(x) + f(-x) = 0$

(iii) Even functions are symmetric about *y*-axis.

(iv) Odd functions are symmetric in diagonally opposite quadrants.

(vi) Every function can be expressed as a sum of an even and an odd function, in the following way ($x, -x \in$ domain (*f*)).

$$f(x) = \left(\frac{f(x) + f(-x)}{2}\right) + \left(\frac{f(x) - f(-x)}{2}\right) \quad \Rightarrow \quad f(x) = h(x) + g(x)$$

Here $h(x)$ is an even function and $g(x)$ is an odd function.

RELATIONS AND FUNCTIONS

RELATIONS AND FUNCTIONS

Example: **Determine the nature of the following functions for even and odd:**

(i) $f(x) = x\left(\dfrac{a^x - 1}{a^x + 1}\right)$ **(ii)** $f(x) = x^2 - |x|$

***Solution*:** (i) We have $f(x) = x\left(\dfrac{a^x - 1}{a^x + 1}\right)$

$$\therefore \quad f(-x) = -x\left(\frac{a^{-x} - 1}{a^{-x} + 1}\right) = -x\left(\frac{\frac{1}{a^x} - 1}{\frac{1}{a^x} + 1}\right) = -x\left(\frac{1 - a^x}{1 + a^x}\right) = x\left(\frac{a^x - 1}{a^x + 1}\right) = f(x)$$

So, $f(x)$ is an even function.

(ii) $f(x) = x^2 - |x| \Rightarrow f(-x) = (-x)^2 - |-x| = x^2 - |x| = f(x)$.

So, $f(x)$ is an even function.
