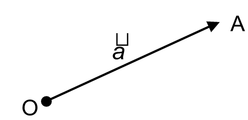
LESSON 11

VECTOR ALGEBRA

1. VECTORS AND SCALARS

Physical quantities, like displacements, velocity, acceleration etc. which have not only a value to signify how much (quantity wise) they are; – this aspect is called the magnitude, – but in addition have a directional element in them, are known as vectors.

A vector may be defined in terms of such a line segment and we write $\vec{a} = \overrightarrow{AB}$.



Vectors are generally printed in bold faced type (in printing). When writing in the manuscript form the notation $\vec{a}, \overrightarrow{AB}$ may be used.

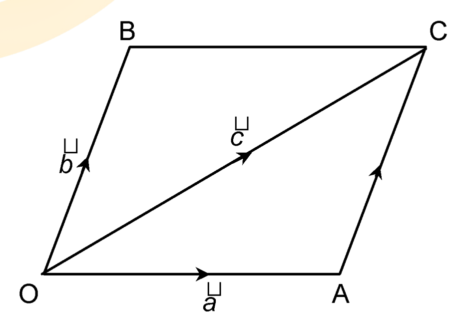
A vector, thus, is geometrically represented by a directed line segment. It is said that $\vec{a}$ is equivalently $\overrightarrow{OA}$ and they are vectorially indistinguishable.

2. LAW OF ADDITION OF VECTORS

The parallelogram law of addition is

$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$$

$$(\vec{a} + \vec{b} = \vec{c})$$



The triangle law is

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$

$$(\vec{a} + \vec{b} = \vec{c})$$

This addition operation, cumulatively, may be had for more than two vectors; and we have

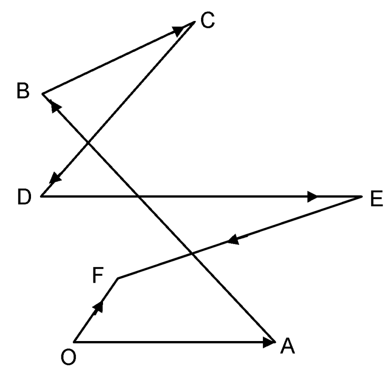
$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) = \vec{a} + \vec{b} + \vec{c}$$

Vector algebra

For addition of more than two vectors we have a polygon laws of vectors addition which is just an extension of triangle law.

$$\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} = \overrightarrow{OF}$$

As a result if terminus of last vector coincides with the initial point of the first vector, then the sum of vectors is a null vector (a vector with zero magnitude).



Example: **If the vectors $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a regular hexagon, express the other sides as vectors in terms of $\vec{a}$ and $\vec{b}$.**

Solution: *ABCDEF* is a regular hexagon.

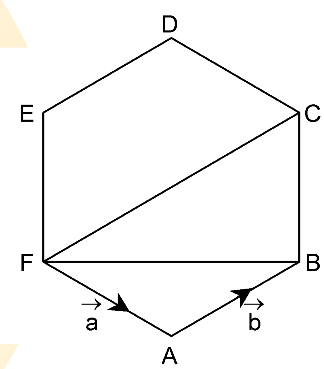
Let $\overrightarrow{FA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$.

$\overrightarrow{FB} = \overrightarrow{FA} + \overrightarrow{AB} = \vec{a} + \vec{b}$

$\overrightarrow{FC} = 2\vec{b}$ ($\overrightarrow{FC}$ is parallel to $\overrightarrow{AB}$ and lengthwise doubled)

$\therefore\ \overrightarrow{BC} = \overrightarrow{FC} - \overrightarrow{FB} = 2\vec{b} - \vec{a} - \vec{b} = \vec{b} - \vec{a}$

$\overrightarrow{CD} = -\vec{a}$; $\overrightarrow{DE} = -\vec{b}$; $\overrightarrow{EF} = \vec{a} - \vec{b}$.



3. MAGNITUDE AND MODULUS OF A VECTOR

The magnitude or modulus of a vector $\vec{a}$ refers to its absolute value and is denoted by $|\vec{a}|$. A vector whose modulus is one unit is called a unit vector, and a vector whose modulus is zero is called a zero vector or a null vector. Such a vector has its length zero, and is therefore geometrically represented by a pair of coincident points.

For two vectors $\vec{a}$ and $\vec{b}$, $||\vec{a}| - |\vec{b}|| \le |\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$.

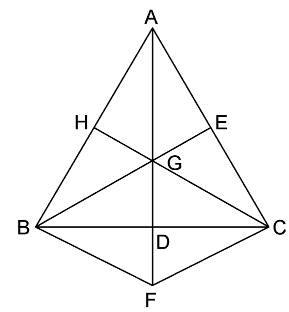
Vector algebra

Example: **If G be the centroid of a triangle ABC, show that $\vec{GA} + \vec{GB} + \vec{GC} = \vec{0}$; and conversely, if $\vec{GA} + \vec{GB} + \vec{GC} = \vec{0}$, then G is the centroid of the triangle ABC.**

Solution: **Necessary part**

Take G as the centroid.

Let the parallelogram GCFB be completed.



$\overrightarrow{GB}+\overrightarrow{GC}=\overrightarrow{GF}$ (parallelogram law of addition of vectors)

$= 2\overrightarrow{GD}$

$= -\overrightarrow{GA}$ ($\because$ $\overrightarrow{GA}$ is oppositely directed to $\overrightarrow{GD}$ and lengthwise doubled)

$\therefore$ $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=0.$

Conversely: Assume that $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=0.$

Let G be joined to the midpoint D of BC and produced to $\vec{F}$

$\overrightarrow{GC}+\overrightarrow{GB}=\overrightarrow{GF}=2\overrightarrow{GD}$

$\therefore$ $2\overrightarrow{GD}+\overrightarrow{GA}=0$ ... (i)

This means that $\overrightarrow{GD}$ and $\overrightarrow{GA}$ have the same directions. Already GD is the join of G to the midpoint of BC. Hence, AGD is a continuous line. So AD is the median. From (i), it is also seen that $\left|\frac{AG}{GD}\right|=\frac{2}{1}$. G is the point of trisection of the median. Hence G is the centroid.

4. MULTIPLICATION OF A VECTOR BY A SCALAR

When a vector is multiplied by a scalar number, its magnitude gets multiplied but direction wise there is no change. Thus $k\vec{a}$ is a vector in the same direction of $\vec{a}$ but magnitude made k times. Thus if in the direction of $\vec{a}$, a unit vector is usually represented as $\hat{a}$ then $\vec{a}=|\vec{a}|\,\hat{a}$.

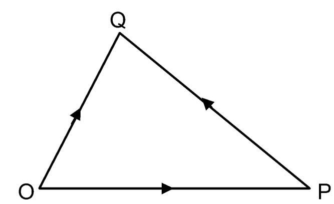
Thus any vector = (its magnitude) unit vector in that direction. It may be also said that $\vec{a}$ and $\hat{a}$ which are direction wise same, are collinear.

5. POSITION VECTOR OF A POINT

Vector algebra

The position vector $\vec{r}$ of any point P with respect to the origin of reference O is a vector $\overrightarrow{OP}$.

For any two points P and Q in the space, the vector $\overrightarrow{PQ}$ can be expressed in terms of their position vectors (p.v.) as



$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

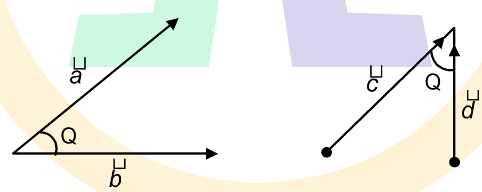
Example: **If $\vec{a}$ and $\vec{b}$ are position vectors of A and B respectively, find the position vector of a point C in AB produced such that $\overrightarrow{AC} = 3\overrightarrow{AB}$.**

Solution: $\overrightarrow{AC} = 3\overrightarrow{AB}$ $\Rightarrow$ $\vec{c} - \vec{a} = 3\,(\vec{b} - \vec{a})$ where $\vec{c}$ is the position vector of point C

$\Rightarrow$ $\vec{c} = \vec{a} + 3\vec{b} - 3\vec{a}$ $\Rightarrow \vec{c} = 3\vec{b} - 2\vec{a}$.

6. ANGLE BETWEEN TWO VECTORS

It is defined as the smaller angle formed when the initial points or the terminal points of two vectors are brought together. Angle between two vectors lies in the interval [0, π].



Example: **If angle between equal vectors $\vec{a}$ and $\vec{b}$ is θ, then find the angle between $\vec{a}$ and $\vec{a} + \vec{b}$.**

Solution: The angle between $\vec{a}$ and $\vec{a} + \vec{b}$ is θ/2 as the diagonal of rhombus bisect the angle between two sides.

7. PARALLEL VECTORS

Two vectors are parallel if they have same direction. They are also known as like vectors. Non-parallel vectors are known as unlike vectors.

Example: **What is the unit vector parallel to $\vec{a} = 3\hat{i} + 4\hat{j} - 2\hat{k}$? What vector should be added to $\vec{a}$ so that the resultant is the unit vector $\hat{i}$?**

Solution: $\vec{a} = 3\hat{i} + 4\hat{j} - 2\hat{k}$ so that $|\vec{a}| = \sqrt{9+16+4} = \sqrt{29}$. Unit vector parallel to $\vec{a}$ is $(3\hat{i} + 4\hat{j} - 2\hat{k})/\sqrt{29}$.

If $\vec{b}$ be added to $\vec{a}$ so that we have

$\vec{a} + \vec{b} = \hat{i}$, then $\vec{b} = \hat{i} - \vec{a}$

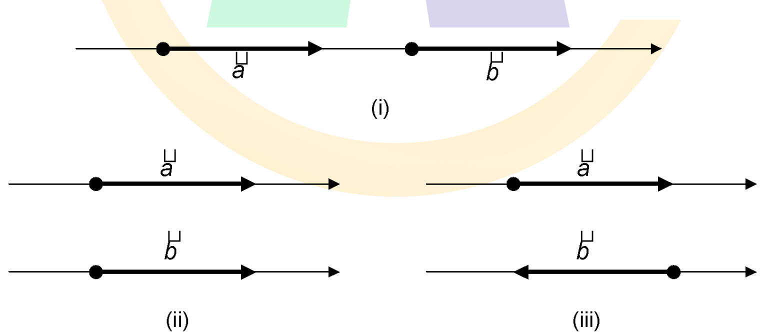
$= \vec{i} - (3\hat{i} + 4\hat{j} - 2\hat{k})$

$= -2\hat{i} - 4\hat{j} + 2\hat{k}$.

8. COLLINEAR VECTORS

Two vectors $\vec{a}$ and $\vec{b}$ are said to be collinear if they are supported on same or parallel lines. Here if their line of support is parallel, vectors may be parallel or anti parallel.

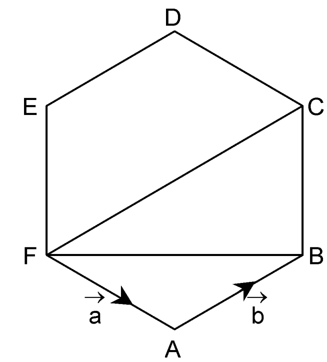
i.e., $\vec{a}$ and $\vec{b}$ can be geometrically represented as



For such vectors $\vec{b} = \lambda\vec{a}$ for some constant λ.

Example: **If the vectors $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a regular hexagon. How many sets of collinear vectors can be found, from the sides of hexagon ?**

Solution: ABCDEF is a regular hexagon.



Let $\vec{FA} = \vec{a}$ and $\vec{AB} = \vec{b}$.

$\vec{FB} = \vec{FA} + \vec{AB} = \vec{a} + \vec{b}$

$\vec{FC} = 2\vec{b}$ ($\vec{FC}$ is parallel to $\vec{AB}$ and lengthwise doubled)

$\therefore\ \vec{BC} = \vec{FC} - \vec{FB} = 2\vec{b} - \vec{a} - \vec{b} = \vec{b} - \vec{a}$

$\vec{CD} = -\vec{a}$; $\vec{DE} = -\vec{b}$; $\vec{EF} = \vec{a} - \vec{b}$.

Thus three sets can be found as: $\vec{AB}$ and $\vec{DE}$; BC and $\vec{EF}$; $\vec{CD}$ and $\vec{FA}$.

9. COPLANAR VECTORS

A set of vectors is said to be coplanar if they lie in same plane, or are all parallel to the same plane. Three vectors $\bar{a}$, $\vec{b}$ and $\vec{c}$ are coplanar if their exist a relationship of the form $\vec{c} = \lambda_1 a + \lambda_2 \vec{b}$ for some scalars λ_1 and λ_2.

Any two non-zero vectors which are non-collinear will constitute a plane. Their sum or difference also lies in the same plane.

Example: **Examine whether $3\vec{a} - 7\vec{b} - 4\vec{c}$; $3\vec{a} - 2\vec{b} + \vec{c}$; $\vec{a} + \vec{b} + 2\vec{c}$ are coplanar.**

Solution: Let $\vec{p} = 3\vec{a} - 7\vec{b} - 4\vec{c}$

$\vec{q} = 3\vec{a} - 2\vec{b} + \vec{c}$

$\vec{r} = \vec{a} + \vec{b} + 2\vec{c}$

Consider

$\vec{p} - 2\vec{q} + 3\vec{r} = 3\vec{a} - 7\vec{b} - 4\vec{c} - 2(3\vec{a} - 2\vec{b} + \vec{c}) + 3(\vec{a} + \vec{b} + 2\vec{c})$

$= \vec{a}\ (3 - 6 + 3) + \vec{b}\ (-7 + 4 + 3) + \vec{c}\ (-4 - 2 + 6) = 0$

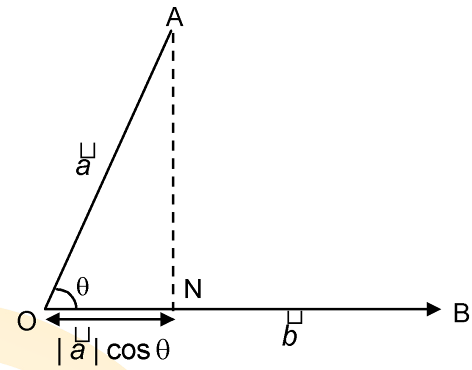
$\therefore\ \vec{p}, \vec{q}, \vec{r}$ are coplanar.

10. PRODUCT OF TWO VECTORS

1. THE SCALAR PRODUCT OR DOT PRODUCT

Let $\vec{a}$ and $\vec{b}$ be any two vectors, forming between the two, an angle θ $(0 \le \theta \le \pi)$.

Then the scalar product or dot product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \cdot \vec{b}$ (Read $\vec{a}$ dot $\vec{b}$) and in value $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$



Geometrically, it represents projection of a vector on the other.

Properties of the Scalar product

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (Dot product is commutative)
2. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ (Dot product is distributive)
3. $\vec{a} \cdot \vec{b} = 0$ either when $|\vec{a}| = 0$ or when $|\vec{b}| = 0$ or when the vectors $\vec{a}$ and $\vec{b}$ are orthogonal. Thus for any two perpendicular vectors the dot product vanishes.
4. $\cos\theta = \left| \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \right|$ where θ is the acute angle made by $\vec{a}$ with $\vec{b}$.
5. $\vec{a} \cdot \vec{a} = \vec{a}^{\,2} = |\vec{a}|^2$

Vector algebra

6. For the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = \hat{i}\cdot\hat{k} = \hat{k}\,.\,\hat{i} = \hat{j}\cdot\hat{i} = \hat{i}\cdot\hat{j} = 0$$

7. If $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ *and* $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, *then* $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$. so that

$$\cos\theta = \left|\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}\right|$$

8. The projection of $\vec{a}$ on another direction represented by $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

9. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then

$$\vec{a}\cdot\hat{i} = a_1\hat{i}\cdot\hat{i} = a_1$$

Similarly $\vec{a}\cdot\hat{j} = a_2$ *and* $\vec{a}\cdot\hat{k} = a_3$ *so that*

$$\vec{a} = (\vec{a}\cdot\hat{i})\,\hat{i} + (\vec{a}\cdot\hat{j})\,\hat{j} + (\vec{a}\cdot\hat{k})\,\hat{k}$$

10. Work done by a force $\vec{F}$ in a displacement $\overrightarrow{AB}$ when the point of application of the force is displaced from *A* to *B*, is, $\vec{F}\cdot\overrightarrow{AB} = \vec{F}\cdot(\overrightarrow{OB} - \overrightarrow{OA}) = \vec{F}\,.$ (position vector of *B* – position vector of *A*)

Example: **Find the angle between** $\vec{a} = 2\,\hat{i} + 2\,\hat{j} - \hat{k}$ ***and*** $\vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$ **.**

Solution: Angle is θ given by cos θ = $\dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}$

$$= \frac{12 - 6 - 2}{\sqrt{9}\,\sqrt{49}} = \frac{4}{21}$$

2. THE VECTOR (OR CROSS) PRODUCT OF TWO VECTORS

Let $\vec{a}$ and $\vec{b}$ be any two vectors forming an angle θ ($0 \leq \theta < \pi$). The vector product or cross product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ (Read as $\vec{a}$ cross $\vec{b}$); and this is a vector

Vector algebra

(a) whose magnitude is $|\vec{a}|\,|\vec{b}|\sin\theta$

(b) whose direction is perpendicular to both $\vec{a}$ and $\vec{b}$ such that looked from this direction the rotation from $\vec{a}$ to $\vec{b}$ through an angle < π is anti-clockwise. It is written as $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\cdot\hat{n}$ where $\hat{n}$ is a unit vector in the direction of $\vec{a} \times \vec{b}$ i.e. in the direction perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.

Properties of the vector product

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (vector product is not commutative)

2. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (vector product is distributive)

3. $\vec{a} \times \vec{b} = 0$ either when $|\vec{a}| = 0$ or when $|\vec{b}| = 0$ or when the vectors have the same direction. Thus the vector product between two collinear vectors is zero.

4. A unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $(\vec{a} \times \vec{b})/|\vec{a} \times \vec{b}|$

5. $\vec{a} \times \vec{a} = 0$ for any vector $\vec{a}$

6. For the unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$ taken along the coordinate axes

$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$ while

$\hat{i}\times\hat{j} = -\hat{j}\times\hat{i} = \hat{k};\quad \hat{j}\times\hat{k} = -\hat{k}\times\hat{j} = \hat{i}$ and

$\hat{k}\times\hat{i} = -\hat{i}\times\hat{k} = \hat{j}$

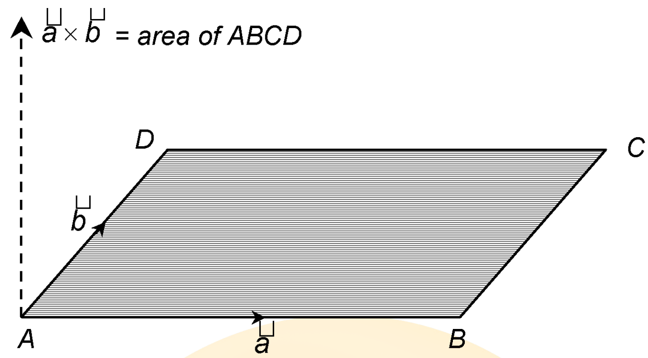
7. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $= \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2)\,\hat{i} + (a_3b_1 - a_1b_3)\,\hat{j} + (a_1b_2 - a_2b_1)\,\hat{k}$$

or in an equivalent determinant form,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

8. $\vec{a} \times \vec{b}$ represents the vector area of the parallelogram whose adjacent sides are represented by $\vec{a}$ and $\vec{b}$.



9. Let $\vec{F}$ be a force directed along a line. Let O be a point (origin). Let $\overrightarrow{OP} = \vec{r}$ be the position vector of any point P on the line of action of $\vec{F}$. Then $\vec{r} \times \vec{F}$ gives the moment of the force $\vec{F}$ about the point O.

10. Let $\vec{\omega}$ be the angular velocity of body rotating about an axis through O. If P be any point of the body with position vector $\overrightarrow{OP} = \vec{r}$, then $\vec{\omega} \times \vec{r}$ gives the velocity vector of P in the rotatory motion about the axis with an angular velocity $\vec{\omega}$.

Example: **For any two vectors $\vec{a}$ and $\vec{b}$ prove that $(\vec{a} \cdot \vec{b})^2 + (\vec{a} \times \vec{b})^2 = |\vec{a}|^2 \, |\vec{b}|^2$**

Solution:

$$\left.\begin{aligned}(\vec{a} \cdot \vec{b})^2 &= |\vec{a}|^2 \, |\vec{b}|^2 \cos^2\theta \\ (\vec{a} \times \vec{b})^2 &= |\vec{a}|^2 \, |\vec{b}|^2 \sin^2\theta\end{aligned}\right\} \textit{where } \theta \text{ is the angle between } \vec{a} \text{ and } \vec{b}.$$

Adding, $(\vec{a} \cdot \vec{b})^2 + (\vec{a} \times \vec{b})^2 = |\vec{a}|^2 \, |\vec{b}|^2 (\cos^2\theta + \sin^2\theta)$

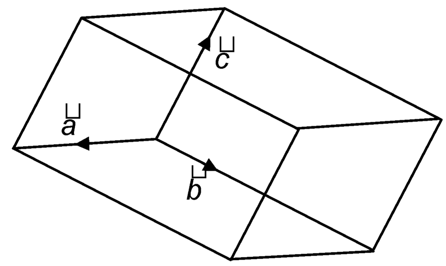
$$= |\vec{a}|^2 \, |\vec{b}|^2$$

$\therefore \vec{c} = (3\hat{i} - 2\hat{j} + 6\hat{k})/7$

11. PRODUCT OF THREE VECTORS

Vector algebra

1. SCALAR TRIPLE PRODUCT



For any two vectors $\vec{b}$ and $\vec{c}$, $\vec{b} \times \vec{c}$ is a vector. This can be scalarly multiplied with a third vector $\vec{a}$ to give the scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})$. This is a scalar whose value is the volume of a box having $\vec{a}$, $\vec{b}$, $\vec{c}$ as coterminous edges. Hence it is also written as [$\vec{a}$ $\vec{b}$ $\vec{c}$] and in this sense is called the box product.

Properties of scalar triple product

1. $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

The value of the scalar triple product is unaltered for a cyclical change of the vectors in the product.

2. $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

The value of the scalar triple product is unaltered for a relative interchange of the positions of the dot and cross.

3. $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ either when $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar (or) when any one vector repeats in the triple product, like $\vec{a}\cdot(\vec{a}\times\vec{b})$, $\vec{a}\cdot(\vec{b}\times\vec{b})$ each one of these is zero.

4. For scalars, *k*, *l* and *m*

$$k\vec{a}\cdot(l\vec{b}\times m\vec{c}) = k\,l\,m\,\vec{a}\cdot(\vec{b}\times\vec{c})$$

5. In particular $\hat{i}.(\hat{j}\times\hat{k}) = 1$

6. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$,

$$\vec{a}\,.\,\vec{b}\times\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and hence the vanishing of this determinant is the condition for the coplanarity of the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ taken in terms of the $\hat{i}, \hat{j}$ and $\hat{k}$ components as shown above.

Example: **Find λ if $\lambda\hat{i} + \hat{j} + 2\hat{k}$; $\hat{i} + \lambda\hat{j} - \hat{k}$ and $2\hat{i} - \hat{j} + \lambda\hat{k}$ are coplanar.**

Solution: The condition for coplanarity is $\begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$

i.e., $\lambda(\lambda^2 - 1) - 1(\lambda + 2) + 2(-1 - 2\lambda) = 0$

$$\lambda^3 - 6\lambda - 4 = 0$$

By inspection it is seen that $\lambda = -2$ is a root.

$\therefore\ \lambda^3 - 6\lambda - 4 = (\lambda + 2)(\lambda^2 - 2\lambda - 2)$

and $\lambda^2 - 2\lambda - 2 = 0$ for $\lambda = 1 \pm \sqrt{3}$

The required value of λ are

$\lambda_1 = -2;\ \lambda_2 = 1 + \sqrt{3};\ \lambda_3 = 1 - \sqrt{3}$

2. VECTOR TRIPLE PRODUCT

For three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ a product of the form $\vec{a} \times (\vec{b} \times \vec{c})$ or $(\vec{a} \times \vec{b}) \times \vec{c}$ is called a vector triple product.

This is a vector, and the value depends upon the placement of the brackets. In fact $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector in the plane of $\vec{b}$ and $\vec{c}$ (the two placed in the brackets).

In value

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$; and

$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a}$

Example: **Show that $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$**

Solution: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$\vec{b} \times (\vec{c} \times \vec{a}) = (\vec{b} \cdot \vec{a})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a}$

$\vec{c} \times (\vec{a} \times \vec{b}) = (\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}$

Adding the three results,

$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$

$= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{c} \cdot \vec{a})\vec{b} + (\vec{b} \cdot \vec{a})\vec{c} - (\vec{a} \cdot \vec{b})\vec{c} + (\vec{c} \cdot \vec{b})\vec{a} - (\vec{b} \cdot \vec{c})\vec{a} = 0$

12. SOME APPLICATIONS OF VECTORS

1. BISECTOR OF AN ANGLE

If $\vec{a}$ and $\vec{b}$ are unit vectors along the sides of an angle, then $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are the vectors along internal and external bisectors of angle respectively.

The bisector of angles between any two vectors $\vec{a}$ and $\vec{b}$ is given by the vector $\vec{r} = \lambda\left(\frac{\vec{a}}{|\vec{a}|} \pm \frac{\vec{b}}{|\vec{b}|}\right)$ where $\lambda \in R$.

Example: **Given $\vec{a}$ and $\vec{b}$ are two non-zero vectors, then $\vec{a}$ and $\vec{b}$ make equal angles with $\vec{c}$ if $\vec{c} = \frac{|\vec{b}|}{|\vec{a}|+|\vec{b}|}\vec{a} + \frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}\vec{b}$.**

Solution: It is evident that $\vec{c}$ is along the bisector of angle between the vectors $\vec{a}$ and $\vec{b}$.

$\Rightarrow \quad \vec{c} = \lambda\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right) = \lambda\left(\frac{|\vec{b}|\vec{a} + |\vec{a}|\vec{b}}{|\vec{a}||\vec{b}|}\right)$

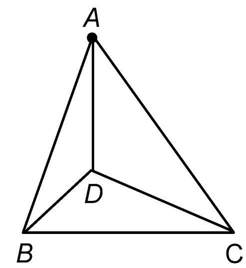
If we consider $\lambda = \frac{|\vec{a}||\vec{b}|}{|\vec{a}|+|\vec{b}|}$

We get $\vec{c} = \frac{|\vec{b}|}{|\vec{a}|+|\vec{b}|}\vec{a} + \frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}\vec{b}$

2. VOLUME OF A TETRAHEDRON

Volume of a tetrahedron *ABCD*

$$= \frac{1}{6}\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]$$



Example: **Find the volume of tetrahedron with one of the vertex at origin and the other 3 at points *A*(3, 4, 2), *B*(0, 4, 1) and *C*(1, 0, 0).**

Solution: Volume $= \frac{1}{6}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right| = \frac{1}{6}\begin{vmatrix} 3 & 4 & 2 \\ 0 & 4 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \frac{2}{3}$ cubic units.

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3. RECIPROCAL SYSTEM OF VECTORS

Let $\vec{a}, \vec{b}, \vec{c}$ be a system of non-coplanar vector, then the system of vectors $\vec{a}', \vec{b}', \vec{c}'$ which satisfy $\vec{a}.\vec{a}', \vec{b}.\vec{b}', \vec{c}.\vec{c}' = 1$ and $\vec{a}.\vec{b}', \vec{a}.\vec{c}', \vec{b}.\vec{a}' = \vec{c}.\vec{a}'$ = $\vec{c}.\vec{b}' = 0$ is called a reciprocal system of vectors. Based on these condition $\vec{a}', b', c'$ are expressed as

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]},\quad \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]},\quad \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Also, $[\vec{a}\,\vec{b}\,\vec{c}]\ [\vec{a}', \vec{b}', \vec{c}'] = 1$

Example: **If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero, non-coplanar vectors then prove that $\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}' = 0$**

Solution: Considering $\vec{a}\times\vec{a}' = \frac{\vec{a}\times(\vec{b}\times\vec{c})}{[\vec{a}\times\vec{b}\times\vec{c}]} = \frac{(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}}{[\vec{a}\times\vec{b}\times\vec{c}]}$

Thus, L.H.S

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}\left((\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a} + (\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\vec{a}\right) = 0$$

13. APPLICATIONS OF VECTOR PRODUCT IN MECHANICS

1. Work done by a force

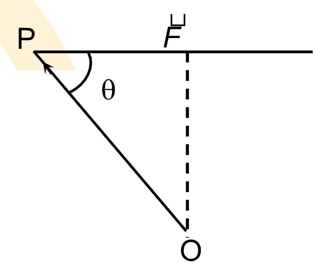
Work done (W) = (Magnitude of force in direction of displacement) × (distance moved)

$$= \left(|\vec{F}|\cos\theta\right)\left(|\vec{d}|\right) = \vec{F}.\vec{d}$$

- The work done by a force is a scalar quantity.
- If a number of faces are acting on a body, then the sum of the works done by the separate forces is equal to the work done by the resultant force.

2. Moment of a force

(a) **About a point:**



Let a force $\vec{F}$ be applied at a point P. The moment of force $\vec{F}$ about a point O is defined as

$$\vec{M} = \overrightarrow{OP} \times \vec{F}$$

- Moment of force about a point is vector quantity.
- Moment is independent of selection of point P, infact P can be any point on the line of action of action of force $\vec{F}$.
- If several forces are acting through the point P, then the vector sum of the moments of the separate forces about O is equal to the moment of their resultant force about O.
- The moment of $\vec{F}$ about a point O measures the amount of $\vec{F}$ to turn the body about point O. If tendency of rotation is in anticlockwise direction, the moment is positive, otherwise it is negative.

(b) **About a line:**

Let $\vec{F}$ be any given force, acting at a point P and L be any directed line segment. The moment of force $\vec{F}$ about line L is defined as

$$M_a = \left(\overrightarrow{OP} \times \vec{F}\right).\hat{a}$$

where $\hat{a}$ is a unit vector in the direction of line and O is any point on the line.

- Moment about a line is a scalar quantity.

- Moment of $\vec{F}$ about the line L is the projection along L, of the vector moment of the force $\vec{F}$ about any point on the L.

Example: **Find the work done by the force $\vec{F} = \hat{i} + \hat{j} + 2\hat{k}$ acting on a particle, if the particle is displaced from the point with position vector $\hat{i} + 2\hat{j} + 2\hat{k}$ to the point with position vector $2\hat{i} + 3\hat{j} + 3\hat{k}$.**

Solution: Here, $\vec{F} = \hat{i} + \hat{j} + 2\hat{k}$ and displacement, $\vec{d} = (2\hat{i} + 3\hat{j} + 3\hat{k}) - (\hat{i} + 2\hat{j} + 2\hat{k}) = \hat{i} + \hat{j} + \hat{k}$

$\therefore$ Work done = $\vec{F}.\vec{d} = (\hat{i} + \hat{j} + 2\hat{k}).(\hat{i} + \hat{j} + \hat{k})$

= $(1)(1) + (1)(1) + (2)(1) = 1 + 1 + 2 = 4$ units