

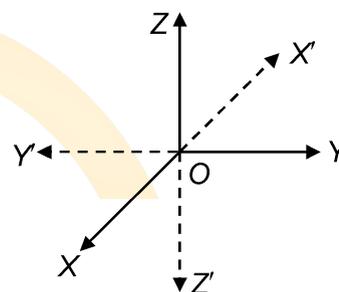
### LESSON 12

### THREE DIMENSIONAL GEOMETRY

Just like a point is specified in two-dimensions with respect to two perpendicular lines, a point in three-dimensional space is specified with respect to three mutually perpendicular lines  $OX$ ,  $OY$  and  $OZ$  called as reference or coordinate axes. Every point in space is uniquely expressed as  $P(x, y, z)$ .

The coordinate system normally used is called 'the right-handed rectangular Cartesian coordinate system.

The planes containing the lines  $X'OX$ ,  $Y'OY$  and  $Z'OZ$  in pairs, determine three mutually perpendicular planes  $XOY$ ,  $YOZ$  and  $ZOX$ .



#### 1. DISTANCE FORMULA

The distance between two points  $P$  and  $Q$  having coordinates  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  is given by

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Distance of a point  $P(x, y, z)$  from origin is given by  $\sqrt{x^2 + y^2 + z^2}$ .

#### 2. SECTION FORMULA

The coordinates of the point  $R$  which divides the line joining two points  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  in the ratio  $m : n$  are given by

$$\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}$$

The division may be internal or external. For external division we take the ratio as negative e.g., 5 : -3 or -5 : 3.

As a special case, the mid-point of  $PQ$  has coordinates  $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

**Example:** Find the ratio in which the line joining the points (1, 2, 3) and (-3, 4, -5) is divided by the  $xy$ -plane. Also find the coordinates of the point of division.

**Solution:** Suppose the line joining the points  $P(1, 2, 3)$  and  $Q(-3, 4, -5)$  is divided by the  $xy$ -plane at a point  $R$  in the ratio  $\lambda : 1$ . Then the coordinates of  $R$  are

$$\left(\frac{-3\lambda + 1}{\lambda + 1}, \frac{4\lambda + 2}{\lambda + 1}, \frac{-5\lambda + 3}{\lambda + 1}\right)$$

Since  $R$  lies on  $xy$ -plane i.e.,  $z = 0$ , therefore

$$\frac{-5\lambda + 3}{\lambda + 1} = 0 \Rightarrow \lambda = \frac{3}{5}$$

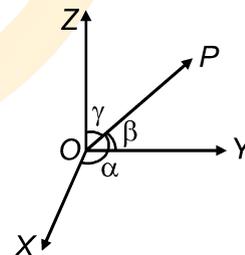
So, the required ratio is  $\frac{3}{5} : 1$  or  $3 : 5$ . Putting  $\lambda = \frac{3}{5}$  in (i), we obtain the coordinates of

$$R \text{ as } \left(\frac{-1}{2}, \frac{1}{4}, 0\right)$$

### 3. DIRECTION COSINES AND DIRECTION RATIOS

#### 1. DIRECTION COSINES

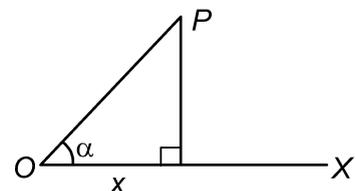
If the position vector of a point  $P$  i.e.,  $\vec{OP}$  makes angles  $\alpha$ ,  $\beta$  and  $\gamma$  with the positive direction of  $x$ ,  $y$  and  $z$  axis respectively, then  $\cos\alpha$ ,  $\cos\beta$  and  $\cos\gamma$  are called its direction cosines. They are also denoted by  $l$ ,  $m$  and  $n$  respectively.



i.e.,  $l = \cos\alpha$ ,  $m = \cos\beta$ ,  $n = \cos\gamma$ .

It can be seen from the figure  $\cos\alpha = \frac{x}{OP}$

Similarly,  $\cos\beta = \frac{y}{OP}$  and  $\cos\gamma = \frac{z}{OP}$



Where  $OP$  is the modulus of positive vector of  $P$ .

Clearly,  $OP = \sqrt{x^2 + y^2 + z^2}$

so,  $l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$$= \frac{x^2 + y^2 + z^2}{OP^2} = 1$$

$\therefore$  if  $\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then  $\hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$

### 2. DIRECTION RATIOS

If  $a, b, c$  three numbers such that  $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$

where  $l, m, n$  are direction cosines of a vector  $\vec{r}$ , then  $a, b, c$  are known as direction numbers or direction ratios of  $\vec{r}$ .

e.g., if  $\hat{r} = 2\hat{i} - 3\hat{j} + 10\hat{k}$

then its direction ratios are 2, -3 and 10 or 4, -6 and 20 or any positive multiple of the components or direction cosines of  $\vec{r}$ .

Two vectors having direction ratios  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$  are parallel if  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ .  
They are perpendicular if  $a_1a_2 + b_1b_2 + c_1c_2 = 0$ .

**Example:** A vector  $\vec{r}$  has length 21 and direction ratios 2, -3, 6. Find the vector  $\vec{r}$ .

**Solution:** The direction cosines of  $\vec{r}$  are

$$\pm \frac{2}{\sqrt{2^2 + (-3)^2 + 6^2}}, \pm \frac{-3}{\sqrt{2^2 + (-3)^2 + 6^2}}, \pm \frac{6}{\sqrt{2^2 + (-3)^2 + 6^2}}$$

Since  $\vec{r}$  makes an acute angle with x-axis, therefore  $\cos \alpha > 0$  i.e.,  $l > 0$ .

So, direction cosines of  $\vec{r}$  are  $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$

$\therefore \vec{r} = 21 \left( \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k} \right)$  [using  $\vec{r} = |\vec{r}| (l\hat{i} + m\hat{j} + n\hat{k})$ ]

or  $\vec{r} = 6\hat{i} - 9\hat{j} + 18\hat{k}$

So, components of  $\vec{r}$  along ox, oy and oz are  $6\hat{i}, -9\hat{j}$  and  $18\hat{k}$  respectively.

### 4. EQUATION OF A STRAIGHTLINE IN SPACE

A straight line in space is specified basically in two ways viz., a line passing through a given point and parallel to a given vector and a line passing through two given points.

#### 1. VECTOR EQUATION OF A LINE PASSING THROUGH A GIVEN POINT AND PARALLEL TO A GIVEN VECTOR

From  $\Delta OAP$ ,  $\vec{OP} = \vec{OA} + \vec{AP}$

$$\Rightarrow \vec{r} = \vec{a} + \lambda \vec{b}$$

#### 2. CARTESIAN EQUATION OF A LINE PASSING THROUGH A GIVEN POINT AND GIVEN DIRECTION RATIOS

Let  $A(a_1, a_2, a_3)$  be the fixed point and the line has direction ratios  $b_1, b_2, b_3$ .

Taking  $\vec{r}$  as  $x\hat{i} + y\hat{j} + z\hat{k}$  in the vector equation we see that

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} = \lambda$$

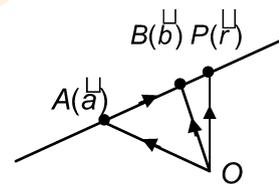
This is the Cartesian equation of the line also called symmetrical form of line. Any point on this line can be taken as

$$(a_1 + b_1\lambda, a_2 + b_2\lambda, a_3 + b_3\lambda)$$

$b_1, b_2, b_3$  can also be replaced by the direction cosines  $l, m, n$  of vector  $\vec{b}$ .

#### 3. VECTOR EQUATION OF A LINE PASSING THROUGH TWO GIVEN POINTS

Let  $A$  and  $B$  be two fixed points having position vectors  $\vec{a}$  and  $\vec{b}$ .  $P$  is a variable point on the line.



From  $\Delta OPA$  again,  $\vec{OP} = \vec{OA} + \vec{AP}$

$$\Rightarrow \vec{OP} = \vec{OA} + \lambda(\vec{AB})$$

$$\Rightarrow \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

This is the required equation.

### 4. CARTESIAN EQUATION OF A LINE PASSING THROUGH TWO GIVEN POINTS

If coordinates of  $A$  and  $B$  are  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ , the Cartesian equation is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

**Example:** Find the vector equation of a line passing through a point with position vector  $2\hat{i} - \hat{j} + \hat{k}$ , and parallel to the line joining the points  $-\hat{i} + 4\hat{j} + \hat{k}$  and  $\hat{i} + 2\hat{j} + 2\hat{k}$ . Also, find the Cartesian equivalent of this equation.

**Solution:** Let  $A, B, C$  be the points with position vectors  $2\hat{i} + \hat{j} + \hat{k}, -\hat{i} + 4\hat{j} + \hat{k}$  and  $\hat{i} + 2\hat{j} + 2\hat{k}$  respectively.

We have to find the equation of a line passing through the point  $A$  and parallel to  $\overrightarrow{BC}$ .

Now,  $\overrightarrow{BC}$  = Position vector of  $C$  - Position vector of  $B$

$$= (\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + 4\hat{j} + \hat{k}) = 2\hat{i} - 2\hat{j} + \hat{k}$$

We know that the equation of a line passing through a point  $\vec{a}$  and parallel to  $\vec{b}$  is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

Here,  $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$

So, the equation of the required line

$$\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k}) \quad \dots(i)$$

Reduction to Cartesian form putting  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ , we obtain

$$x\hat{i} + y\hat{j} + z\hat{k} = (2 + 2\lambda)\hat{i} + (-1 - 2\lambda)\hat{j} + (1 + \lambda)\hat{k}$$

$$\Rightarrow x = 2 + 2\lambda, y = -1 - 2\lambda, z = 1 + \lambda$$

$$\Rightarrow \frac{x - 2}{2} = \frac{y + 1}{-2} = \frac{z - 1}{1},$$

which is the Cartesian equivalent of (i).

### 5. ANGLE BETWEEN TWO LINES

If two lines are parallel to vectors  $\vec{b}_1$  and  $\vec{b}_2$ , the angle between them is given by

$$\cos \theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

**Example:** Find the angle between the lines  $\frac{x-2}{3} = \frac{y+1}{-2}, z=2$  and  $\frac{x-1}{1} = \frac{2y+3}{3} = \frac{z+5}{2}$ .

**Solution:** The given equations are not in the standard form. The equations of the given lines can be written as

$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-2}{0}$$

and,  $\frac{x-1}{1} = \frac{y+3/2}{3/2} = \frac{z+5}{2}$

Let  $\vec{b}_1$  and  $\vec{b}_2$  be vectors parallel to (i) and (ii) respectively, then,

$$\vec{b}_1 = 3\hat{i} - 2\hat{j} + 0\hat{k} \quad \text{and} \quad \vec{b}_2 = \hat{i} + \frac{3}{2}\hat{j} + 2\hat{k}$$

If  $\theta$  is the angle between the given lines, then

$$\cos \theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|} = \frac{(3)(1) + (-2)(3/2) + (0)(2)}{\sqrt{3^2 + (-2)^2} \sqrt{1^2 + (3/2)^2 + 2^2}} = 0$$

$$\Rightarrow \theta = \pi/2.$$

## 5. INTERSECTION OF TWO LINES

Two lines in space can have the following three positions:

- (i) They are parallel
- (ii) They are intersecting
- (iii) They are neither intersecting nor parallel. Such lines are called skew lines.

**Example:** Show that the lines  $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$  and  $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$  intersect. Find the point of intersection.

**Solution:** The position vectors of arbitrary points on the given lines are  $(\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$   
 $= (3\lambda + 1)\hat{i} + (1 - \lambda)\hat{j} - \hat{k}$  and  $(4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k}) = (2\mu + 4)\hat{i} + 0\hat{j} + (3\mu - 1)\hat{k}$  respectively.

If the lines intersect, then they have a common point. So, for some values of  $\lambda$  and  $\mu$ , we must have

$$(3\lambda + 1)\hat{i} + (1 - \lambda)\hat{j} - \hat{k} = (2\mu + 4)\hat{i} + 0\hat{j} + (3\mu - 1)\hat{k}$$

$$\Rightarrow 3\lambda + 1 = 2\mu + 4, 1 - \lambda = 0 \text{ and } -1 = 3\mu - 1$$

Solving last two of these three equations, we get  $\lambda = 1$  and  $\mu = 0$ . These values of  $\lambda$  and  $\mu$  satisfy the first equation. So, the given lines intersect. Putting  $\lambda = 1$  in first line, we get

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + (3\hat{i} - \hat{j}) = 4\hat{i} + 0\hat{j} - \hat{k} \text{ as the position vector of the point of intersection.}$$

Thus, the coordinates of the point of intersection are  $(4, 0, -1)$ .

## 6. SHORTEST DISTANCE BETWEEN TWO LINES

If  $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$  are two skew lines, the shortest distance between them is the perpendicular distance.

$$d = \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}$$

More appropriately,

$$d = \left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right| = \left| \frac{[\vec{b}_1, \vec{b}_2, (\vec{a}_2 - \vec{a}_1)]}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Clearly two lines intersect if  $[\vec{b}_1, \vec{b}_2, (\vec{a}_2 - \vec{a}_1)] = 0$

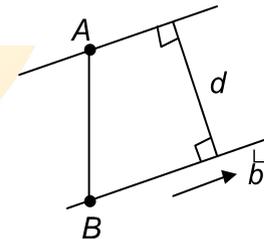
If the lines are parallel,

$$\text{i.e., } \vec{r} = \vec{a}_1 + \lambda\vec{b}$$

$$\text{and } \vec{r} = \vec{a}_2 + \lambda\vec{b}$$

the formula to calculate shortest distance becomes

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$$



## 7. PERPENDICULAR DISTANCE OF A POINT FROM A LINE

The perpendicular distance can be obtained using vector form as well as Cartesian form of the line.

$$\Rightarrow d = \left| \frac{(\vec{p} - \vec{a}) \times \vec{b}}{|\vec{b}|} \right|$$

**Example:** Find the foot of the perpendicular drawn from the point  $2\hat{i} - \hat{j} + 5\hat{k}$  to the line  $\vec{r} = (11\hat{i} - 2\hat{j} - 8\hat{k}) + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k})$ . Also find the length of the perpendicular.

**Solution:** Let  $L$  be the foot of the perpendicular drawn from  $P(2\hat{i} - \hat{j} + 5\hat{k})$  on the line

$$\vec{r} = 11\hat{i} - 2\hat{j} - 8\hat{k} + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k})$$

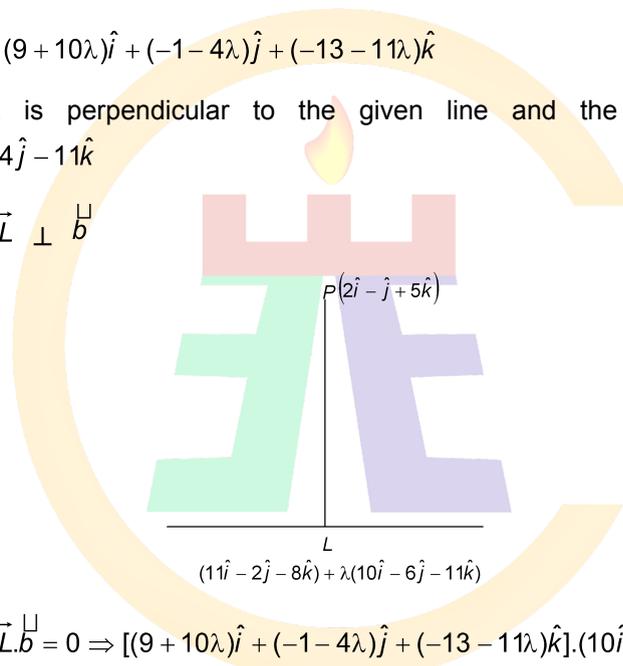
Let the position vector of  $L$  be

$$11\hat{i} - 2\hat{j} - 8\hat{k} + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k}) = (11 + 10\lambda)\hat{i} + (-2 - 4\lambda)\hat{j} + (-8 - 11\lambda)\hat{k}. \text{ Then}$$

$$\begin{aligned} \vec{PL} &= \text{position vector of } L - \text{position vector of } P \\ &= [(11 + 10\lambda)\hat{i} + (-2 - 4\lambda)\hat{j} + (-8 - 11\lambda)\hat{k}] - [2\hat{i} - \hat{j} + 5\hat{k}] \\ &= (9 + 10\lambda)\hat{i} + (-1 - 4\lambda)\hat{j} + (-13 - 11\lambda)\hat{k} \end{aligned}$$

Since  $PL$  is perpendicular to the given line and the given line is parallel to  $\vec{b} = 10\hat{i} - 4\hat{j} - 11\hat{k}$

$$\therefore \vec{PL} \perp \vec{b}$$



$$\begin{aligned} \Rightarrow \vec{PL} \cdot \vec{b} &= 0 \Rightarrow [(9 + 10\lambda)\hat{i} + (-1 - 4\lambda)\hat{j} + (-13 - 11\lambda)\hat{k}] \cdot (10\hat{i} - 4\hat{j} - 11\hat{k}) = 0 \\ \Rightarrow 10(9 + 10\lambda) - 4(-1 - 4\lambda) - 11(-13 - 11\lambda) &= 0 \\ \Rightarrow 90 + 100\lambda + 4 + 16\lambda + 143 + 121\lambda &= 0 \\ \Rightarrow 237\lambda = -237 \Rightarrow \lambda &= -1 \end{aligned}$$

Putting the value of  $\lambda$ , we obtain the position vector of  $L$  as  $\hat{i} + 2\hat{j} + 3\hat{k}$

$$\text{Now, } \vec{PL} = (\hat{i} + 2\hat{j} + 3\hat{k}) - (2\hat{i} - \hat{j} + 5\hat{k}) = -\hat{i} + 3\hat{j} - 2\hat{k}$$

Hence, length of the perpendicular from  $P$  on the given line  $= |\vec{PL}| = \sqrt{1 + 9 + 4} = \sqrt{14}$ .

### 8. EQUATION OF PLANE IN VARIOUS FORMS

A plane can be defined as a surface on which if any two points are arbitrarily chosen, the line segment joining those points lies completely on the surface.

### 1. EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND NORMAL TO A GIVEN VECTOR

$$\Rightarrow (\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

Cartesian form of equation of plane is  $ax + by + cz + d = 0$

where  $a, b, c$  denote the direction ratios of the normal vector.



**Example:** Find the equation in Cartesian form of the plane passing through the point (3, -3, 1) and normal to the line joining the points (3, 4, -1) and (2, -1, 5).

**Solution:** We know that the vector equation of a plane passing through a point having position vector

$\vec{a}$  and normal to  $\vec{n}$  is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0 \quad \text{or} \quad \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

Since the given plane passes through the point (3, -3, 1) and is normal to the line joining A (3, 4, -1) and B (2, -1, 5). Therefore

$$\begin{aligned} \vec{a} &= 3\hat{i} - 3\hat{j} + \hat{k} \quad \text{and} \quad \vec{n} = \vec{AB} = P.V. \text{ of B} - P.V. \text{ of A} \\ &= (2\hat{i} - \hat{j} + 5\hat{k}) - (3\hat{i} + 4\hat{j} - \hat{k}) = -\hat{i} - 5\hat{j} + 6\hat{k} \end{aligned}$$

Substituting  $\vec{a} = 3\hat{i} - 3\hat{j} + \hat{k}$  and  $\vec{n} = -\hat{i} - 5\hat{j} + 6\hat{k}$  in (i), we obtain

$$\vec{r} \cdot (-\hat{i} - 5\hat{j} + 6\hat{k}) = (3\hat{i} - 3\hat{j} + \hat{k}) \cdot (-\hat{i} - 5\hat{j} + 6\hat{k})$$

or  $\vec{r} \cdot (-\hat{i} - 5\hat{j} + 6\hat{k}) = -3 + 15 + 6$  or  $\vec{r} \cdot (-\hat{i} - 5\hat{j} + 6\hat{k}) = 18$

This is the vector equation of the required plane. The Cartesian equation is

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (-\hat{i} - 5\hat{j} + 6\hat{k}) = 18$$

$$\Rightarrow -x - 5y + 6z = 18 \quad \text{or} \quad x + 5y - 6z + 18 = 0$$

### 2. EQUATION OF A PLANE NORMAL TO A GIVEN VECTOR AND AT A GIVEN DISTANCE FROM ORIGIN

Let a plane be perpendicular to a unit vector  $\hat{n}$  and at a perpendicular distance  $d$  from origin.

$$\Rightarrow \vec{r} \cdot \hat{n} = d$$

### 3. EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND PARALLEL TO TWO GIVEN VECTORS

$$\vec{r} = \vec{a} + \lambda \vec{b} + \mu \vec{c}$$

Here  $\lambda$  and  $\mu$  are arbitrary scalars.

This form is also called the parametric form of the plane. It can also be written in the non-parametric form as

$$(\vec{r} - \vec{a}) \cdot \vec{b} \times \vec{c} = 0$$

or  $[\vec{r} \ \vec{b} \ \vec{c}] = [\vec{a} \ \vec{b} \ \vec{c}]$



**Example:** Find the vector equation of the following plane in scalar product form:

$$\vec{r} = (\hat{i} - \hat{j}) + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$$

**Solution:** We know that the equation  $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$  represents a plane passing through a point having position vector  $\vec{a}$  and parallel to vectors  $\vec{b}$  and  $\vec{c}$ . Here

$$\vec{a} = \hat{i} - \hat{j}, \vec{b} = \hat{i} + \hat{j} + \hat{k} \text{ and } \hat{i} - 2\hat{j} + 3\hat{k}$$

The given planes is perpendicular to the vector

$$\vec{n} = \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\hat{i} - 2\hat{j} - 3\hat{k}$$

So, the vector equation of the plane in scalar product form is

$$\begin{aligned} \vec{r} \cdot \vec{n} &= \vec{a} \cdot \vec{n} \\ \Rightarrow \vec{r} \cdot (5\hat{i} - 2\hat{j} - 3\hat{k}) &= (\hat{i} - \hat{j}) \cdot (5\hat{i} - 2\hat{j} - 3\hat{k}) \\ \Rightarrow \vec{r} \cdot (5\hat{i} - 2\hat{j} - 3\hat{k}) &= (5 + 2 + 0) \text{ or } \vec{r} \cdot (5\hat{i} - 2\hat{j} - 3\hat{k}) = 7 \end{aligned}$$

#### 4. EQUATION OF A PLANE PASSING THROUGH THREE GIVEN POINTS

Then  $\vec{AB}$  and  $\vec{BC}$  lie in the plane. So, as in the previous article the equation of plane becomes

$$\begin{aligned} \vec{r} &= \vec{a} + \lambda\vec{AB} + \mu\vec{AC} \quad \text{or} \quad \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) + \mu(\vec{c} - \vec{a}) \\ \Rightarrow \vec{r} &= (1 - \lambda - \mu)\vec{a} + \lambda\vec{b} + \mu\vec{c} \end{aligned}$$

#### 5. INTERCEPT FORM OF A PLANE

The equation of a plane intercepting the coordinate axes at  $(a, 0, 0)$ ,  $(0, b, 0)$  and  $(0, 0, c)$  is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

**Example:** Write the equation of the plane whose intercepts on the coordinate axes are  $-4$ ,  $2$  and  $3$ .

**Solution:** We know that the equation of a plane whose intercepts on the coordinate axes are  $a$ ,  $b$  and

$c$  respectively, is  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ .

Here,  $a = -4$ ,  $b = 2$ , and  $c = 3$ . So, the equation of the required plane is

$$\frac{x}{-4} + \frac{y}{2} + \frac{z}{3} = 1 \quad \text{or} \quad -3x + 6y + 4z = 12.$$

## 9. ANGLE BETWEEN TWO PLANES

The angle between two planes is defined as the angle between their normals. If  $\vec{n}_1$  and  $\vec{n}_2$  are the normals and  $\theta$  is the angle then

$$\cos \theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| |\vec{n}_2|}$$

Obviously, two planes are perpendicular if  $\vec{n}_1 \cdot \vec{n}_2 = 0$ .

They are parallel if  $\vec{n}_1 = \lambda \vec{n}_2$  where  $\lambda$  is a scalar.

**Example:** Find the angle between the planes  $x + y + 2z = 9$  and  $2x - y + z = 15$ .

**Solution:** We know that the angle between the planes  $a_1x + b_1y + c_1z + d_1 = 0$  and  $a_2x + b_2y + c_2z + d_2 = 0$  is given by

$$\cos \theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Therefore, angle between  $x + y + 2z = 9$  and  $2x - y + z = 15$  is given by

$$\cos \theta = \frac{(1)(2) + (1)(-1) + (2)(1)}{\sqrt{1^2 + 1^2 + 2^2} \sqrt{2^2 + (-1)^2 + 1^2}} = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

## 10. FAMILY OF PLANES

### 1. PLANE PARALLEL TO A GIVEN PLANE

Since parallel planes have the same normal vector, so equation of a plane parallel to  $\vec{r} \cdot \hat{n} = d_1$  is of the form  $\vec{r} \cdot \hat{n} = d_2$ , where  $d_2$  is determined by the given conditions.

In Cartesian form, if  $ax + by + cz + d = 0$  be the given plane then the plane parallel to this plane is  $ax + by + cz + k = 0$ .

### 2. PLANE PASSING THROUGH INTERSECTION OF TWO PLANES

Two planes intersect in a line if they are not parallel. Any plane through the line of intersection of two planes can be written as

$$(\vec{r} \cdot \vec{n}_1 - d_1) + \lambda(\vec{r} \cdot \vec{n}_2 - d_2) = 0 \quad (\text{vector form})$$

or  $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0 \quad (\text{Cartesian form})$

where  $\lambda$  is a real number.

**Example:** Find the equation of the plane through the point  $(1, 4, -2)$  and parallel to the plane  $-2x + y - 3z = 7$ .

**Solution:** Let the equation of a plane parallel to the plane  $-2x + y - 3z = 7$  be

$$-2x + y - 3z + k = 0 \quad \dots(i)$$

This passes through  $(1, 4, -2)$ , therefore

$$(-2)(1) + 4 - 3(-2) + k = 0$$

$$\Rightarrow -2 + 4 + 6 + k = 0$$

$$\Rightarrow k = -8.$$

Putting  $k = -8$  in (i), we obtain

$$-2x + y - 3z - 8 = 0$$

or  $-2x + y - 3z = 8$

This is the equation of the required plane.

## 11. DISTANCE OF A POINT FROM A PLANE

The perpendicular distance of a point  $P(\vec{p})$  from the plane  $\vec{r} \cdot \vec{n} = d$  is given by  $\frac{|\vec{p} \cdot \vec{n} - d|}{|\vec{n}|}$

In Cartesian form, the perpendicular distance of  $P(x_1, y_1, z_1)$  from the plane  $ax + by + cz + d = 0$  is equal to

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

### 12. DISTANCE BETWEEN PARALLEL PLANES

The distance between  $ax + by + cz + d_1 = 0$  and  $ax + by + cz + d_2 = 0$  is given by

$$\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$



**Example:** Find the distance between the parallel planes  $2x - y + 2z + 3 = 0$  and  $4x - 2y + 4z + 5 = 0$ .

**Solution:** Let  $P(x_1, y_1, z_1)$  be any point on  $2x - y + 2z + 3 = 0$ , then

$$2x_1 - y_1 + 2z_1 + 3 = 0 \quad \dots(i)$$

The length of the perpendicular from  $P(x_1, y_1, z_1)$  to  $4x - 2y + 4z + 5 = 0$  is

$$\frac{|4x_1 - 2y_1 + 4z_1 + 5|}{\sqrt{4^2 + (-2)^2 + 4^2}} = \frac{|2(2x_1 - y_1 + 2z_1) + 5|}{\sqrt{36}} = \frac{|2(-3) + 5|}{6} = \frac{1}{6} \quad \text{[using (i)]}$$

Therefore, the distance between the two given parallel planes is  $\frac{1}{6}$ .

## 13. LINE AND PLANE

### 1. UNSYMMETRICAL FORM OF LINE

The equation of two non-parallel planes taken together is called the unsymmetrical form of line because if a point lies in both these planes it has to lie on the line of intersection of the planes. The following illustration shows how to convert the unsymmetrical form of line to symmetrical form.

**Example:** Reduce in symmetrical form, the equation of the line of intersection two planes  $x - y + 2z = 5$ ,  $3x + y + z = 6$ .

**Solution:** Let  $a, b, c$  be the direction ratios of the required line.

Since the required line lies in both the given planes, we must have

$$a - b + 2c = 0 \quad \text{and} \quad 3a + b + c = 0$$

Solving these two equations by cross-multiplication, we get

$$\frac{a}{-1-2} = \frac{b}{6-1} = \frac{c}{1+3} \quad \text{or} \quad \frac{a}{-3} = \frac{b}{5} = \frac{c}{4}$$

In order to find a point on the required line, we put  $z = 0$  in the two given equations to obtain

$$x - y = 5, \quad 3x + y = 6$$

Solving these two equations, we obtain  $x = \frac{11}{4}, y = \frac{-9}{4}$ .

Therefore, coordinates of a point on the required line are  $(\frac{11}{4}, \frac{-9}{4}, 0)$ . Hence, the equation of the required line is

$$\frac{x - \frac{11}{4}}{-3} = \frac{y - \left(-\frac{9}{4}\right)}{5} = \frac{z - 0}{4} \quad \text{or} \quad \frac{4x - 11}{-12} = \frac{4y + 9}{20} = \frac{z - 0}{4}$$

$$\text{or} \quad \frac{4x - 11}{-3} = \frac{4y + 9}{5} = \frac{z - 0}{1}$$

### 2. ANGLE BETWEEN A LINE AND A PLANE

if the plane is  $ax + by + cz + d = 0$  and line is  $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$

then 
$$\sin \theta = \frac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2} \sqrt{l^2 + m^2 + n^2}}$$

condition  $al + bm + cn = 0$  line parallel to plane

line perpendicular to plane  $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$

**Example:** Find the angle between the line  $\frac{x+1}{3} = \frac{y-1}{2} = \frac{z-2}{4}$  and the plane  $2x + y - 3z + 4 = 0$ .

**Solution:** The given line is parallel to the vector  $\vec{b} = 3\hat{i} + 2\hat{j} + 4\hat{k}$  and the given plane is normal to the vector  $\vec{n} = 2\hat{i} + \hat{j} - 3\hat{k}$ . Therefore, the angle  $\theta$  between the given line and given plane is given by

$$\begin{aligned} \sin \theta &= \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}| |\vec{n}|} = \frac{(3\hat{i} + 2\hat{j} + 4\hat{k}) \cdot (2\hat{i} + \hat{j} - 3\hat{k})}{\sqrt{3^2 + 2^2 + 4^2} \sqrt{2^2 + 1^2 + (-3)^2}} \\ &\Rightarrow \sin \theta = \frac{6 + 2 - 12}{\sqrt{29} \sqrt{14}} = \frac{-4}{\sqrt{406}} \\ &\Rightarrow \theta = \sin^{-1} \left( \frac{-4}{\sqrt{406}} \right) \end{aligned}$$