INVERSE TRIGONOMETRIC FUNCTIONS

1. DOMAIN AND RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

The equations tan x = y and x = $\tan^{-1} y$ are not identical because the former associates many values of x to a single value of y while the latter associates a single x to a particular value of y .In the same way, the remaining five inverse trigonometric functions are also defined .To assign a unique angle to a particular value of trigonometric ratio, we introduce a term called 'principal range'. The principal ranges of all the inverse trigonometric functions have been fixed. *e.g.,*

Principal range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. *i.e.,* We have to search for an angle in this interval only.

$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ only,

although $\sin\frac{5\pi}{6} = \frac{1}{2}$, $\sin\frac{13\pi}{6} = \frac{1}{2}$,

etc. $\left(\text{note that } \sin^{-1}\frac{1}{2} \neq \frac{1}{\sin\frac{1}{2}}\right)$.

The principal range of inverse trigonometric functions is the most important thing in this lesson. All formulae and problems are linked in some way or the other to that only.

We list below the domain and principal ranges of all the six inverse trigonometric functions.

Function	Domain (values of *x*)	Principal Range (values of *y*)
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$y = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$

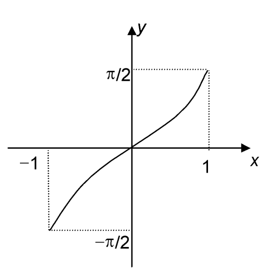
$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6},\ \text{cosec}^{-1}(1) = \frac{\pi}{2},\ \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \frac{2\pi}{3},\ etc.$$

Example: **Evaluate: $\tan^{-1}(-1)$.**

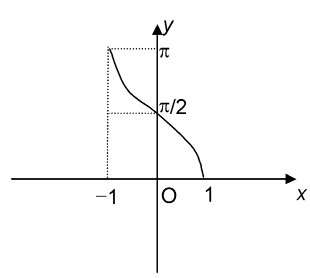
Solution : $\tan\left(\frac{-\pi}{4}\right) = -1$,

$\therefore\ \tan^{-1}(-1) = -\frac{\pi}{4}\ \left\{\because\ \frac{-\pi}{4} \in \text{range of } \tan^{-1} x\right\}$

2. GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS



$y = \sin^{-1}x$



$y = \cos^{-1}x$

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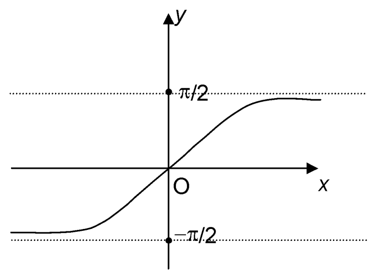
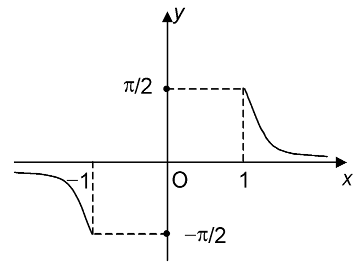
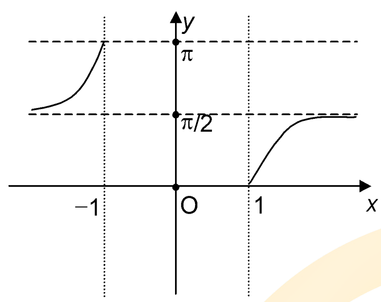


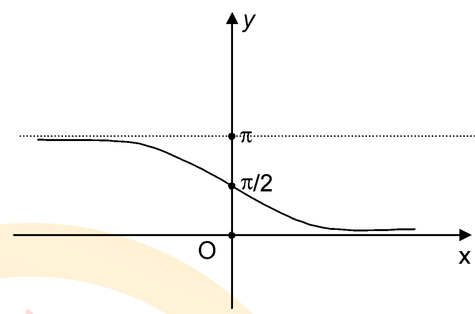
y* = tan^{-1}*x



y* = cosec^{-1}*x



y* = sec^{-1}*x



y* = cot^{-1}*x

3. SOME BASIC RESULTS

We have the following relations :

· $\sin^{-1}\sin\theta = \theta$	if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
· $\cos^{-1}\cos\theta = \theta$	if $0 \le \theta \le \pi$
· $\tan^{-1}\tan\theta = \theta$	if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
· $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$	if $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$
· $\sec^{-1}(\sec\theta) = \theta$	if $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$
· $\cot^{-1}(\cot\theta) = \theta$	if $0 < \theta < \pi$

Similarly,

· $\sin(\sin^{-1} x) = x$	if $\lvert x\rvert \le 1$
· $\cos(\cos^{-1} x) = x$	if $\lvert x\rvert \le 1$
· $\tan(\tan^{-1} x) = x$	if $x \in R$
· $\text{cosec}(\text{cosec}^{-1} x) = x$	if $\lvert x\rvert \ge 1$
· $\sec(\sec^{-1} x) = x$	if $\lvert x\rvert \ge 1$

· $\cot(\cot^{-1} x) = x$ if $x \in R$

Example: **Find the angle** $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

Solution: Let

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \theta$$

$$\tan^{-1}\tan\left(\pi - \frac{\pi}{4}\right) = \theta$$

$$\tan^{-1}\left(-\tan\frac{\pi}{4}\right) = \theta$$

$\Rightarrow$ $$-\tan^{-1}\tan\frac{\pi}{4} = \theta \quad \left[\text{As } \tan^{-1}\tan\theta = \theta, \text{if } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$

$\Rightarrow$ $$-\frac{\pi}{4} = \theta$$

Hence to solve this type of problem, the procedure is to add and subtract π till it belongs to the principal value range of respective inverse trigonometric function.

4. FORMULAE IN INVERSE TRIGONOMETRY

$$2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \text{only if } -1 \le x \le 1$$

$$= \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right) \quad \text{if } x > 1$$

$$2\tan^{-1} x = -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right) \quad \text{if } \quad x < -1$$

Consider few important results:

1. $\bullet\ \sin^{-1}(-x) = -\sin^{-1} x \quad \forall\ x \in [-1, 1]$

$\bullet\ \cos^{-1}(-x) = \pi - \cos^{-1} x \quad \forall\ x \in [-1, 1]$

$\bullet\ \tan^{-1}(-x) = -\tan^{-1} x \quad \forall\ x \in R$

$\bullet\ \cot^{-1}(-x) = \pi - \cot^{-1} x \quad \forall\ x \in R$

$\bullet\ \sec^{-1}(-x) = \pi - \sec^{-1} x \quad \forall\ x \in (-\infty, -1] \cup [1, \infty)$

$\bullet\ \mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1} x \quad \forall\ x \in (-\infty, -1] \cup [1, \infty)$

2. $\tan^{-1}(1/x) = \begin{cases} \cot^{-1} x & \forall\ x > 0 \\ -\pi + \cot^{-1} x & \forall\ x < 0 \end{cases}$

Similarly, within the domain of their definitions,

$$\tan^{-1} x = \begin{cases} \cot^{-1} \dfrac{1}{x} & , \ x > 0 \\ \cot^{-1}\left(\dfrac{1}{x}\right) - \pi & , \ x < 0 \end{cases}$$

$$\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x, \quad \forall x \in (-\infty, -1] \cup [1, \infty)$$

$$\sin^{-1} x = \operatorname{cosec}^{-1} \frac{1}{x}, \ \forall\ x \in [-1,\ 0) \cup (0,\ 1]$$

$$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x, \quad \forall x \in (-\infty, -1] \cup [1, \infty)$$

$$\cos^{-1} x = \sec^{-1} \frac{1}{x}, \ \forall\ x \in [-1,\ 0) \cup (0,\ 1]$$

Example: **Obtain the value of $\cos^{-1}\left(-\frac{3}{5}\right) + \sin^{-1}\left(-\frac{5}{13}\right)$ in terms of $\cos^{-1}$ function.**

Solution :

$$\cos^{-1}\left(-\frac{3}{5}\right) + \sin^{-1}\left(-\frac{5}{13}\right)$$

$$= \pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13}\right)$$

Let $\sin^{-1}\frac{4}{5} = \alpha \Rightarrow \sin\alpha = \frac{4}{5}$

$\sin^{-1}\frac{5}{13} = \beta \Rightarrow \sin\beta = \frac{5}{13}$

consider $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$$= \frac{3}{5}\cdot\frac{12}{13} - \frac{4}{5}\cdot\frac{5}{13} = \frac{16}{65}$$

$\Rightarrow \alpha + \beta = \cos^{-1}\frac{16}{65}$ ($\alpha, \beta \in$ quadrant 1)

$\therefore$ Given quantity $= \pi - \cos^{-1}\frac{16}{65} = \cos^{-1}\left(-\frac{16}{65}\right)$

3.

- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $-1 \le x \le 1$
- $\cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}$, $x \in R$
- $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$, $-\infty < x \le -1$ or $1 \le x < \infty$

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Example**:** **Evaluate cos $[2\cos^{-1}x + \sin^{-1}x]$ at $x = \frac{1}{5}$.**

Solution: $2\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2} + \cos^{-1}x = \frac{\pi}{2} + \theta$ where $\cos\theta = x$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta = -\sqrt{1-x^2} = -\sqrt{1-\frac{1}{25}} = \frac{-2\sqrt{6}}{5}$$

Example: **If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ prove that $x^2 + y^2 + z^2 + 2xyz = 1$.**

Solution: Given: $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\Rightarrow$ $\cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z = \cos^{-1}(-z)$

$\Rightarrow$ $\cos[\cos^{-1}x + \cos^{-1}y] = \cos[\cos^{-1}(-z)]$

Let $\cos^{-1}x = A;$ $\cos^{-1}y = B$

$\therefore$ $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\therefore$ $\cos(A + B) = xy - \sqrt{1-x^2}\sqrt{1-y^2}$

$\therefore$ $(A + B) = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$

$\Rightarrow$ $\cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\} = \cos^{-1}(-z)$ $\Rightarrow$ $xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$\Rightarrow$ $(xy + z)^2 = (1 - x^2)(1 - y^2)$ $\Rightarrow$ $x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow$ $x^2 + y^2 + z^2 + 2xyz = 1.$

Hence proved.