MATRICES

1. DEFINITION OF A MATRIX

A system of '*mn*' numbers (real or complex) arranged in a rectangular array of *m* rows and *n* columns is called a matrix. This system can be arranged in any of the following patterns.

$$(\quad);\ [\quad];\ \|\quad\|$$

In general a_{ij} represent the element (or entry) of i^{th} row and j^{th} column, so the matrix can be represented as (a_{ij}) or $[a_{ij}]$ or $||a_{ij}||$

2. ORDER OF A MATRIX

If any matrix *A* contains '*m*' rows and '*n*' columns then $m \times n$ is termed as order of matrix.

Order is generally written as suffix of the array.

Now any matrix of order $m \times n$ will have the notation $[a_{ij}]_{m\times n}$.

i.e. $A = [a_{ij}]_{m\times n}$ or $(a_{ij})_{m\times n}$ or $\|a_{ij}\|_{m\times n}$

it is obvious that $1 \le i \le m$ and $1 \le j \le n$

3. TYPES OF MATRIX

The elements which appear in the rectangular array are known as entries ; depending upon these entries, matrices are of following types:

1. ROW MATRIX

A single row matrix is called a row matrix or a row vector.

e.g. the matrix $[a_{11}\ a_{12}\ \ldots\ldots\ a_{1n}]$ is a $1 \times n$ row matrix.

2. COLUMN MATRIX

A single column matrix is called a column matrix or a column vector.

e.g. the matrix $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \ldots \\ \ldots \\ a_{mj} \end{bmatrix}$ is a $m \times 1$ column matrix.

3. SQUARE MATRIX

If $m = n$, i.e. if the number of rows and columns of a matrix are equal say n, then it is called a square matrix of order n.

4. NULL (or zero) MATRIX

If all the elements of a matrix are equal to zero, then it is called a null matrix and is denoted by $O_{m \times n}$ or O.

5. DIAGONAL MATRIX

A square matrix in which all its elements are zero except those in the leading diagonal, is called a diagonal matrix. Thus in a diagonal matrix $a_{ij} = 0$ if $i \neq j$.

The diagonal matrices of order 2 and 3 are as follows:

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}.$$

The elements a_{ij} of a matrix for which $i = j$ are called the diagonal elements of a matrix and the diagonal along which all these elements lie is called the principal diagonal or the diagonal of the matrix.

6. SCALAR MATRIX

A square matrix in which all the diagonal elements are equal and all other elements equal to zero is called a scalar matrix.

i.e. in a scalar matrix $a_{ij} = k$, for i = j and $a_{ij} = 0$ for i ≠ j. Thus $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ is a scalar matrix.

7. UNIT MATRIX OR IDENTITY MATRIX

A square matrix in which all its diagonal elements are equal to 1 and all other elements equal to zero is called a unit matrix or identity matrix.

e.g. a unit (or identity) matrix of order 2 and 3 are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

8. NEGATIVE OF A MATRIX

Let A = $[a_{ij}]_{m \times n}$ be a matrix. Then the negative of the matrix A is defined as the matrix $[-a_{ij}]_{m \times n}$ and is denoted by −A.

4. EQUALITY OF MATRICES

Two matrices A and B are said to be equal, written as $A = B$, if,

(i) they both are of the same order i.e. have the same number of rows and columns, and

(ii) the elements in the corresponding places of the two matrices are the same.

5. ADDITON AND SUBTRACTION OF MATRICES

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same type $m \times n$. Then their sum (or difference) $A + B$ (or $A - B$) is defined as another matrix of the same type, say $C = [c_{ij}]$ such that any element of C is the sum (or difference) of the corresponding elements of A and B.

$\therefore \quad C = A \pm B = [a_{ij} \pm b_{ij}]$

Example: $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 7 & 3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$.

Solution: Here both A and B are 2×3 matrices

$$\therefore \quad A+B=\begin{bmatrix}1+7 & 2+3 & 4+2\\ 0+5 & 5+1 & 3+9\end{bmatrix}=\begin{bmatrix}8 & 5 & 6\\ 5 & 6 & 12\end{bmatrix}$$

and
$$A-B=\begin{bmatrix}1-7 & 2-3 & 4-2\\ 0-5 & 5-1 & 3-9\end{bmatrix}=\begin{bmatrix}-6 & -1 & 2\\ -5 & 4 & -6\end{bmatrix}$$

PROPERTIES OF MATRIX ADDITION

1. $A + B = B + A$

2. $A + (B + C) = (A + B) + C$

3. $k(A + B) = kA + kB$ here k is any scalar.

4. $A + O = O + A = A$, here O {null matrix} will be additive identity.

5. If A be a given matrix then the matrix $-A$ is the additive inverse of A for $A + (-A) =$ null matrix O.

6. If A, B and C be three matrices of the same type
then $A + B = A + C \Rightarrow B = C$ (Left Cancellation Law)
and $B + A = C + A \Rightarrow B = C$ (Right Cancellation Law)

6. MULTIPLICATION OF A MATRIX BY A SCALAR

Let $A = [a_{ij}]_{m\times n}$ be a matrix and k a scalar. Then the matrix obtained by multiplying each element of matrix A by k is called the scalar multiple of A and is denoted by $k\,A$.

PROPERTIES

- If k_1 and k_2 are scalars and A be a matrix, then $(k_1 + k_2)\,A = k_1A + k_2A$.
- If k_1 and k_2 are scalars and A be a matrix, then $k_1(k_2\,A) = (k_1k_2)A$.
- If A and B are two matrices of the same order and k, a scalar, then $k(A + B) = kA + kB$.
  i.e. the scalar multiplication of matrices distributes over the addition of matrices.
- If A is any matrix and k be a scalar, then $(-k)A = -(kA) = k(-A)$.

7. MULTIPLICATION OF TWO MATRICES

Let $A = [a_{ij}]$ be $m \times p$ matrix and $B = [b_{ij}]$ be $p \times n$ matrix. These matrices A and B are such that the number of columns of A are the same as the number of rows of B each being equal to p. Then the product AB (in the order it is written) will be a matrix $C = [c_{ij}]$ of the type $m \times n$.

Where c_{ij} will be the element of C occurring in i^{th} row and j^{th} column and it will be row by column product of i^{th} row of A having p columns with j^{th} column of B having p rows, the elements of which are

$a_{i1}\ a_{i2}\ \ldots\ a_{ip}$ and b_{1j}

$a_{i1}\ a_{i2}\ \ldots\ a_{ip}$ and b_{2j}

………………………

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$$\therefore \quad c_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \ldots\ a_{ip}\, b_{pj} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

The summation is to be performed w.*r*.t. repeated suffix k.

Above gives us the particular i-jth element of C which is $m \times n$ type. For getting an element of C occurring in 2nd row and 3rd column we shall put $i = 2$ and $j = 3$.

$$\therefore \quad c_{23} = \sum_{k=1}^{p} a_{2k} b_{k3} = a_{21} b_{13} + a_{22} b_{23} + \ldots\ldots a_{2p} b_{p3}$$

There being m rows in A, i can take values from 1 to m and there being n columns in B, j can take values from 1 to n, and thus we shall get all the mn elements of C.

Again $$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \qquad \ldots(i)$$

Above gives us i-jth element of AB which is of $m \times n$ type having m rows and n columns.

PROPERTIES OF MATRIX MULTIPLICATION

(a) Multiplication of matrices is distributive with respect to addition of matrices

i.e. $A\,(B + C) = AB + AC$.

(b) Matrix multiplication is associative if conformability is assured.

i.e. $A(BC) = (AB)\,C$.

(c) The multiplication of matrices is not always commutative.

i.e. AB is not always equal to BA.

(d) Multiplication of a matrix A by a null matrix conformable with A, will give null-matrix.

i.e. Let $$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 1 & 2 \\ 6 & 4 & 2 \\ 7 & 4 & 6 \end{bmatrix}_{4\times 3} \text{ and } O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times 2} \Rightarrow AO = O$$

Example: **If** $A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ **compute** ***AB*** **and** ***BA*****.**

Solution: Here A is 3 × 3 and B is 3 × 3. Hence both AB and BA are defined and each will be 3 × 3 matrix.

Let $$AB = C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

where C_{ij} means that take the product of ith row of A with jth column of B.

e.g. C_{23} = product of 2nd row of A with 3rd column of B.

i.e. $$[-3 \quad 2 \quad -1]\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = -3.3 + 2.6 - 1.3 = 0$$.

Similarly we can find other elements of C.

We can also say that by the product of first row of A with the three columns of B ; we shall get the three elements of first row of C.

i.e. R_1C_1, R_1C_2, R_1C_3

and similarly take the second row of A and multiply with all the columns of B and you will get the three elements of 2nd row of C i.e. R_2C_1, R_2C_2, R_2C_3 and elements of 3rd row of C will be R_3C_1, R_3C_2, R_3C_3.

$$\therefore \quad AB = \begin{bmatrix} 1.1 - 1.2 + 1.1 & 1.2 - 1.4 + 1.2 & 1.3 - 1.6 + 1.3 \\ -3.1 + 2.2 - 1.1 & -3.2 + 2.4 - 1.2 & -3.3 + 2.6 = 1.3 \\ 2.1 + 1.2 + 0.1 & -2.2 + 1.4 + 0.2 & -2.3 + 1.6 + 0.3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$ (i.e. , null matrix)

Similarly BA can also be computed.

8. OPERATIONS REGARDING MATRICES

TRANSPOSE OF A MATRIX

If A be a given matrix of the type $m \times n$ then the matrix obtained by changing the rows of A into columns and columns of A into rows is called transpose of matrix A and is denoted by A' or A^T. As there are m rows in A therefore there will be m columns in A' and similarly as there are n columns in A there will be n rows in A'.

Properties of transpose

(i) $(A')' = A$

(ii) $(KA)' = KA'$. K being a scalar.

(iii) $(A \pm B)' = A' \pm B'$

(iv) $(AB)' = B' A'$.

9. TYPES OF MATRIX ON THE BASIS OF OPERATIONS

1. SYMMETRIC MATRIX

A square matrix $A = [a_{ij}]$ is said to be symmetric if its $(i, j)^{th}$ element is the same as its (j, i)th element i.e., if $a_{ij} = a_{ji}$ for all i, j.

2. SKEW SYMMETRIC MATRIX

A square matrix $A = [a_{ij}]$ is said to be skew symmetric if the $(i, j)^{th}$ element of A is the negative of the (j, i)th element of A i.e., if $a_{ij} = -a_{ji}$ for all i, j.

10. ADJOINT OF A SQUARE MATRIX

Let $A = [a_{ij}]_{n\times n}$ be any $n \times n$ matrix. The transpose B' of the matrix $B = [C_{ij}]_{n\times n}$, where C_{ij} denotes the cofactor of the element a_{ij} in the determinant $|A|$, is called the adjoint of the matrix A and is denoted by the symbol *adj A*.

Example: **If** $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ **, then find *adj* A.**

Solution: In $|A|$, the cofactor of α is δ and the cofactor of β is −γ. Also the cofactor of γ is −β and the cofactor of δ is α. Therefore the matrix B formed of the cofactor of the elements of $|A|$ is

$$B = \begin{bmatrix} \delta & -\gamma \\ -\beta & \alpha \end{bmatrix}$$

Now Adj A = the transpose of the matrix $B = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$

11. INVERSE OF A MATRIX

Let A be any n-rowed square matrix. Then a matrix B, if it exists, such that $AB = BA = I_n$ is called inverse of A.

The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$.

If A be an invertible matrix, then the inverse of A is $\frac{1}{|A|}$ Adj. A. It is usual to denote the inverse of A by A^{-1}.

Example: **Find the inverse of the matrix** $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution: We have

$$|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 3 & 1 & -1 \end{vmatrix}$$, applying $C_3 \to C_3 - 2C_2$

$= -1\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix}$, expanding the determinant along the first

$= -2$

Since $|A| \neq 0$, therefore A^{-1} exists.

Now the cofactors of the elements of the first row of $|A|$ are $\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$

i.e. , are −1 , 8 , −5 respectively.

The cofactors of the elements of the second row of $|A|$ are $-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$

i.e. , are 1, −6, 3 respectively.

The cofactors of the elements of the third row of $|A|$ are $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$

i.e. , are −1, 2, −1 respectively.

Therefore the Adj. A = the transpose of the matrix B where

$$B = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix} \quad \therefore \text{ Adj. } A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Now $A^{-1} = \frac{1}{|A|}$ Adj. A and here $|A| = -2$.

$$\therefore A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

12. SINGULAR AND NON-SINGULAR MATRICES

A square matrix A is said to be non-singular or singular according as $|A| \neq 0$ or $|A| = 0$.

13. ELEMENTARY OPERATIONS OR ELEMENTARY TRANSFORMATIONS OF A MATRIX

Definition

Any of the following operations is called an **elementary transformation (operation).**

(i) The interchange of any two rows (or columns).

(ii) The multiplication of the elements of any row (or column) by a non-zero number.

(iii) The addition to the elements of any row (or column), the corresponding elements of any other row (or column) multiplied by a non-zero number.

1. The elementary operations of interchange of *i*th row and *j*th row is denoted by $R_i \leftrightarrow R_j$ and interchange of *i*th column and *j*th column is denoted by $C_i \leftrightarrow C_j$.

Example:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$ i.e., interchanging Ist row and 3rd row matrix A becomes the matrix

$$B = \begin{bmatrix} 2 & 0 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

2. The elementary operation of multiplication of the elements of the *i*th row by a non-zero number k is denoted by $R_i \to kR_i$.

Similarly, the multiplication of the elements of the *i*th column by a non-zero number k is denoted by $C_i \to kC_i$.

Example:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$

On multiplying the elements of 3rd column of matrix A by 2, i.e., on applying $C_3 \to 2C_3$, we get the new matrix

$$B = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 3 & 8 \\ 2 & 0 & 10 \end{bmatrix}$$

3. The elementary operation of the addition to the elements of the *i*th row, the corresponding elements of the *j*th row multiplied by a non-zero number *k* is denoted by $R_i \to R_i + kR_j$. Similarly, the elementary operation of the addition to the elements of the *i*th column, the corresponding elements of the *j*th column multiplied by a non-zero number *k* is denoted by $C_i \to C_i + kC_j$.

Example:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 7 & 4 \\ 2 & 4 & 5 \end{bmatrix}$

On applying the elementary operation $C_2 \to C_2 + 2C_1$, matrix *A* becomes the matrix *B*.

- **Equivalent Matrices**

Two matrices *A* and *B* are said to be **equivalent** if one can be obtained from other by applying a finite number of elementary operations on the other matrix. If *A* and *B* are equivalent matrices we write $A \sim B$.

Example:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & 6 \\ 2 & 0 & 10 \end{bmatrix}$

Now $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 2 & 0 & 5 \end{bmatrix}$ [Applying $R_1 \leftrightarrow R_2$]

$\sim \begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & 6 \\ 2 & 0 & 10 \end{bmatrix} = B$ [Applying $C_3 \to 2C_3$]

Here $A \sim B$ as *B* has been obtained from *A* by applying two elementary operations.

- **Elementary Matrix:**

A matrix obtained from unit matrix by a single elementary operation is called an elementary matrix.

Example:

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $[R_1 \to 2R_1]$

is an elementary matrix.

Inverse of a matrix by elementary operations (Elementary operations on matrix equation)

Let A, B and X be three matrices of the same order such that

$$X = AB \qquad \ldots(i)$$

The matrix equation (i) will be also valid if we apply a row operation on matrix X [occurring on the L.H.S. of equation (i)] and the same row operation on matrix A (the first factor of product AB on the matrix on R.H.S.)

Thus on the application of a sequence of row operations on the matrix equation $X = AB$ (these row operations are applied on X and on the first matrix A of product AB simultaneously), the matrix equation is still valid (we assume this fact without proof).

Similarly a sequence of elementary column operations on the matrix equation $X = AB$ can be applied simultaneously on X and on the second matrix B of product AB and the equation will be still valid.

In view of the above mentioned fact, it is clear that we can find the inverse of a matrix A, if it exists, by using either a sequence of elementary row operations or a sequence of elementary column operations but not both simultaneously.

Using row operation

Apply a series of row operation on $A = IA$ till we get $I = BA$

Now by definition of inverse of a matrix $B = A^{-1}$

Using Column operation

Apply a series of column operations on $A = AI$ till we get $I = AB$. By definition of inverse B is inverse of A.

Example: **Obtain the inverse of the matrix using elementary operations** $\boldsymbol{A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}}$.

Solution: **Using row operation**

$A = IA$, i.e., $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ (applying $R_1 \leftrightarrow R_2$)

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$ (applying $R_3 \to R_3 - 3R_1$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$ (applying $R_1 \to R_1 - 2R_2$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$ (applying $R_3 \to R_3 + 5R_2$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$ (applying $R_3 \to \frac{1}{2} R_3$)

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$ (applying $R_1 \to R_1 + R_3$)

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$ (applying $R_2 \to R_2 - 2R_3$)

Hence $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$

14. HOMOGENEOUS LINEAR EQUATIONS

The equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = 0 \ , \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = 0 \ , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = 0 \end{array}\right\} \quad \ldots\text{(i)}$$

represents system of m homogeneous equations in n unknowns x_1, x_2 , ……. , x_n. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ \dots \\ x_n \end{bmatrix}_{n\times 1}, \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ \dots \\ 0 \end{bmatrix}_{m\times 1},$$

where A, X, O are $m \times n$, $n \times 1$, $m \times 1$ matrices respectively. Then obviously we can write the system of equations (i) in the form of a single matrix equation

$$AX = O \qquad \dots\text{(ii)}$$

The matrix A is called the coefficient matrix of the system of equations.

(i) If $|A| = 0$ the system has infinitely many solutions.

(ii) If $|A| \neq 0$ the system has zero solution or trivial solutions.

Example: **Does the following system of equations possess a common non-zero solution?**

$$x + 2y + 3z = 0$$
$$3x + 4y + 4z = 0$$
$$7x + 10y + 12z = 0$$

Solution: Determinant of coefficient matrix is $|A| = -2$ which non-zero

$\therefore$ $x = y = z = 0$ is the only solution.

15. SYSTEM OF LINEAR NON HOMOGENEOUS EQUATIONS

The equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2, \\ &\dots\dots\dots\dots \\ &\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \qquad \dots\text{(i)}$$

be a system of m non-homogeneous equations in n unknowns x_1, x_2, ……., x_n. If we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}_{n\times 1}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}_{m\times 1},$$

where A, X, B are $m \times n$, $n \times 1$, $m \times 1$ matrices respectively the above equations can be written in the form of a single matrix equation $AX = B$.

Any set of values of x_1 , x_2 , ... , x_n which simultaneously satisfy all these equations is called a solution of the system (i). When the system of equations has one or more solutions, the equations are said to be consistent, otherwise they are said to be inconsistent.

If $B \neq 0$ the system (i) is said to be non-homogenous.

(i) If $|A| \neq 0$

the given system has unique solution.

(ii) If $|A| = 0$

$\therefore$ for infinitely many solutions to the system $(adj\ A)B = 0$

Clearly for no solution $(adj\ A)B \neq 0$

Example: **Show that the equations $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$, $6y - 18z + 1 = 0$ are not consistent.**

Solution:

$$\Delta = |A| = \begin{vmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{vmatrix} = 0$$

$$\Delta_1 = \begin{vmatrix} -11 & 6 & 0 \\ -3 & 20 & -6 \\ -1 & 6 & -18 \end{vmatrix} \neq 0 \; ; \quad \Delta_2 = \begin{vmatrix} 2 & -11 & 6 \\ 6 & -3 & 20 \\ 0 & -1 & 6 \end{vmatrix} \neq 0 \; ; \quad \Delta_3 = \begin{vmatrix} 2 & 6 & -11 \\ 6 & 20 & -3 \\ 0 & 6 & -1 \end{vmatrix} \neq 0$$

$\Rightarrow$ the system is inconsistent