e-Edge Education Centre

Determinants

Class XII-EMM-MT-
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LESSON 4

DETERMINANTS

1. DETERMINANTS

1. EVALUATION OF DETERMINANTS

Determinants of second order:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \qquad \text{... (i)}$$

Determinants of third order:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \qquad \text{... (ii)}$$

Note that each term of a second order determinant is the product of two quantities and each term of a third order determinant is the product of three quantities.

2. MINORS

The minor of a_1 in (ii) is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, and the minor of b_2 in (ii) is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

3. COFACTORS

In (ii), the elements a_1, b_1, c_1 are multiplied by

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

These expressions are called the cofactors of the elements a_1, b_1, c_1.

***Example*:** **Evaluate the determinant** $\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 5 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

Solution: Expanding along the second row, we have

$$\Delta = -5\begin{vmatrix}3 & 4\\2 & 3\end{vmatrix} - 2\begin{vmatrix}2 & 4\\1 & 3\end{vmatrix} - 1\begin{vmatrix}2 & 3\\1 & 2\end{vmatrix} = -5(9-8) - 2(6-4) - 1(4-3)$$

$= -5 - 4 - 1 = -10.$

Expanding along the third column, we have

$$\Delta = 4\begin{vmatrix}5 & -2\\1 & 2\end{vmatrix} - 1\begin{vmatrix}2 & 3\\1 & 2\end{vmatrix} + 3\begin{vmatrix}2 & 3\\5 & -2\end{vmatrix} = 4(10+2) - 1(4-3) + 3(-4-15)$$

$= 48 - 1 - 57 = -10.$

2. PROPERTIES OF DETERMINANTS

1. If two rows in a determinant are interchanged, the sign of the determinant changes. For example, by interchanging the two rows of the determinant $\begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix}$, we get the determinant $\begin{vmatrix}a_2 & b_2\\a_1 & b_1\end{vmatrix}$

But we have $\begin{vmatrix}a_2 & b_2\\a_1 & b_1\end{vmatrix} = -\begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix}$

2. If the numbers in one row are added, m times the numbers in another row, the value of the determinant remains unaltered.

For example, $\begin{vmatrix}a_1+ma_2 & b_1+mb_2\\a_2 & b_2\end{vmatrix} = \begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix}$

This rule can be extended to more number of rows for higher order determinants.

3. If rows and columns are interchanged, the value of the determinant remains unaltered.

For example, $\begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix} = \begin{vmatrix}a_1 & a_2\\b_1 & b_2\end{vmatrix}$

Another way of saying this is that it makes no difference if we reflect the numbers of the determinant in the line of the principal diagonal. This means that any statement that can truly be made about rows in particular results (1) and (2) can equally well be made about columns.

4. If all the numbers in any row are zeros, the value of the determinant is zero.

For example, $\begin{vmatrix}a_1 & b_1 & c_1\\0 & 0 & 0\\a_3 & b_3 & c_3\end{vmatrix} = 0$

5. If two rows are identical, the value of the determinant is zero.

For example, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$

6. If the elements of a row are multiplied by any number m, the determinant is multiplied by *m*.

For example, $\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

7. Row-Column Operations : The value of determinant remains unchanged when any row (or column) is multiplied by a number or any expression and then added or subtracted from any other row (or column).

i.e. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ma_2 - na_3 & a_2 & a_3 \\ b_1 + mb_2 - nb_3 & b_2 & b_3 \\ c_1 + mc_2 - nc_3 & c_2 & c_3 \end{vmatrix}$

The above operation is written as $C_1 \to C_1 + mC_2 - nC_3$ means C_1 is replaced by $C_1 + mC_2 - nC_3$.

***Example*:** **Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ = (*a* − *b*) (*b* − *c*) (*c* − *a*).**

***Solution*:** Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

If *b* is put equal to a, two rows are exactly alike.

∴ Δ = 0 when *b* = *a*

∴ (*a* − *b*) is a factor of Δ (This follows from the factor theorem which states that for *f*(*x*), if *f*(*a*) = 0, then (*x* − a) is a factor of *f*(*x*)).

Similarly (*b* − *c*) and (*c* − a) are factors.

Again, Δ is of the third degree in *a*, *b* and *c*.

And we know already three linear factors (*a* − *b*), (*b* − *c*) and (*c* − *a*). If there is another factor, it must be a mere number.

Thus $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ = *N* (*a* − *b*) (*b* − *c*) (*c* − *a*), where *N* is a number.

By equating coefficients of bc^2 on both sides, *N* = 1

∴ Δ = (*a* − *b*) (*b* − *c*) (*c* − *a*).

Alternative method:

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Subtracting the second row from the first and then the third row from the second, we have

$$\Delta = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Now expanding along the first column, we have

$\Delta = (a - b)(b - c)[(b + c) - (a + b)] = (a - b)(b - c)(c - a)$.

3. SUM OF DETERMINANTS

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ be two third order determinants in which corresponding second and third columns are identical.

Then $\Delta_1 + \Delta_2 = \begin{vmatrix} a_1+d_1 & b_1 & c_1 \\ a_2+d_2 & b_2 & c_2 \\ a_3+d_3 & b_3 & c_3 \end{vmatrix}$

This fact is evident if we expand all the three determinants in terms of column 1 and compare the results.

Similarly if $\Delta_3 = \begin{vmatrix} p_1 & q_1 & r_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

then $\Delta_1 + \Delta_3 = \begin{vmatrix} a_1+p_1 & b_1+q_1 & c_1+r_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Here we note that the corresponding second and third rows are identical.

Similarly the determinant $\begin{vmatrix} d_1+e_1+f_1 & d_2+e_2+f_2 & d_3+e_3+f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Can be decomposed into the sum of three determinants

$$\begin{vmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} e_1 & e_2 & e_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

It may be observed that the determinant

$$\begin{vmatrix} a_1+b_1 & c_1+d_1 & e_1+f_1 \\ a_2+b_2 & c_2+d_2 & e_2+f_2 \\ a_3+b_3 & c_3+d_3 & e_3+f_3 \end{vmatrix}$$ can be expressed as sum of $2 \times 2 \times 2 = 8$ determinants.

4. SYSTEM OF LINEAR EQUATIONS

1. THE SYSTEM OF TWO LINEAR EQUATIONS IN TWO UNKNOWNS

Consider the system of two linear equations in two unknowns:

$a_1x + b_1y = c_1$; $a_2x + b_2y = c_2$

Solving the system we get the answer

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}; \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Note: The given equations are consistent and independent if and only if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$.

***Example*:** **Solve the system $4x + y = 13$, $3x - 2y = 7$ using determinants.**

***Solution*:** The solution requires the values of three determinants.

The denominator Δ is formed by writing the coefficients of x and y in order

$$\Delta = \begin{vmatrix} 4 & 1 \\ 3 & -2 \end{vmatrix} = -8 - 3 = -11$$

d_1, the numerator of x is formed by replacing the coefficients of x by the constant terms.

$$d_1 = \begin{vmatrix} 13 & 1 \\ 7 & -2 \end{vmatrix} = -26 - 7 = -33$$

d_2, the numerator of y is formed by replacing the coefficients of y by the constant terms.

$$d_2 = \begin{vmatrix} 4 & 13 \\ 3 & 7 \end{vmatrix} = 28 - 39 = -11$$

Then $x = \frac{d_1}{\Delta} = \frac{-33}{-11} = 3$

and $y = \frac{d_2}{\Delta} = \frac{-11}{-11} = 1$

2. SYSTEM OF THREE EQUATIONS IN TWO UNKNOWNS

The following system of equations

$a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$; $a_3x + b_3y + c_3 = 0$ is consistent

if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Example: **Find those values of *c* for which the equations $2x+3y=3; (c+2)x+(c+4)y=(c+6)$ and $(c+2)^2x+(c+4)^2y=(c+6)^2$ are consistent. Also solve the equations for those values of *c*.**

Solution: The condition for consistency is $\begin{vmatrix} 2 & 3 & 3 \\ c+2 & c+4 & c+6 \\ (c+2)^2 & (c+4)^2 & (c+6)^2 \end{vmatrix} = 0$

i.e. $\begin{vmatrix} -1 & 3 & 0 \\ -2 & c+4 & 2 \\ -2(2c+6) & (c+4)^2 & 2(2c+10) \end{vmatrix} = 0$

i.e. $(-1)\{(c+4)(2c+10)-(c+4)^2\}-3\{-2(2c+10)+2(2c+6)\}=0$

i.e. $c^2+8c+16-2c^2-18c-40+12c+60-12c-36=0$

i.e. $-c^2-10c=0 \Rightarrow$ $c=0$ or $c=-10$

For c = 0,the 3 equation are 2x + 2y = 3

2x + 4y = 6 ; 4x + 16y = 36

and the solution is $x=-3$; $y=3$. For $c=-10$, the equation are 2x + 3y = 3

$-8x-6y=-4 \Rightarrow 4x+3y=2$

$64x-36y=16 \Rightarrow 16x+9y=4$ and the corresponding solution is $x=-\frac{1}{2}; y=\frac{4}{3}$

3. THE SYSTEM OF THREE LINEAR EQUATIONS IN THREE UNKNOWNS

Consider the system of three linear equations in three unknowns:

$a_1x + b_1y + c_1z = p$

$a_2x + b_2y + c_2z = q$

$a_3x + b_3y + c_3z = r$

The solution of the system may be expressed as

$x=\frac{d_1}{\Delta}, y=\frac{d_2}{\Delta}, z=\frac{d_3}{\Delta}$, where

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $d_1 = \begin{vmatrix} p & b_1 & c_1 \\ q & b_2 & c_2 \\ r & b_3 & c_3 \end{vmatrix}$, $d_2 = \begin{vmatrix} a_1 & p & c_1 \\ a_2 & q & c_2 \\ a_3 & r & c_3 \end{vmatrix}$, $d_3 = \begin{vmatrix} a_1 & b_1 & p \\ a_2 & b_2 & q \\ a_3 & b_3 & r \end{vmatrix}$

Note: The determinant Δ is formed by writing the coefficients of *x*, *y*, z in order while the determinant appearing in the numerator for any unknown is obtained from Δ by replacing the column of coefficients of that unknown by the column of constants.

4. CRAMERS RULE

Consider the system of *n* linear equations in *n* unknowns given by

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

$$...$$

$$...$$

$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

Let $D = \begin{vmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ : & & & : \\ a_{n1} & a_{n2} & .... & a_{nn} \end{vmatrix}$

Let D_j be the determinant obtained from D after replacing the j^{th} column by $\begin{vmatrix} b_1 \\ . \\ . \\ . \\ b_n \end{vmatrix}$

Then, if $D \neq 0$, we have $x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, ..., x_n = \frac{D_n}{D}$.

Discuss $D = 0$ cases

Case (i)

If $D = 0$ and the other determinants $D_1 = D_2 = ... = D_n = 0$, then system of equation has infinitely many solutions if all cofactors of $D_1, D_2, ..., D_n$ and D are zero. If any one cofactor of $D_1, D_2, D_3, ..., D_n$ is non zero then system has no solution.

eg. $x + 3y + 2z = 1$; $2x + 6y + 4z = 5$; $3x + 9y + 6z = 9$

Here $D_x = D_y = D_z = D = 0$ yet system has no solution where as

$x + 3y + 2z = 1$; $2x + 6y + 4z = 2$; $3x + 9y + 6z = 3$ has infinitely many solutions.

Case (ii)

If $D = 0$ but any one of the D_1, D_2 .... or D_n is not equal to zero then the system has no solution, hence is inconsistent.

5. THE SYSTEM OF HOMOGENEOUS LINEAR EQUATIONS

Eliminate and non-trivial solution: If the three equations (homogeneous)

$a_1x + b_1y + c_1z = 0; a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$ be considered then there always exists a solution i.e., $x = y = z = 0$. This is called the **Trivial Solution.**

If the three equations are to have a solution other than $x = 0 = y = z$, such a solution is known as **Non-Trivial** solution, then the condition required for the existence of such a solution is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Example: **Let λ and α be real. Find the set of all values of λ for which the system of linear equations:**

$$\lambda x + (\sin\alpha)\, y + (\cos\alpha)\, z = 0$$

$$x + (\cos\alpha)\, y + (\sin\alpha)\, z = 0$$

$$-x + (\sin\alpha)\, y - (\cos\alpha)\, z = 0$$

has a non-trivial solution. For λ = 1 find all the values of α.

Solution: The condition for the existence of non-trivial solution (trivial solution is $x = y = z = 0$) is

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0 \quad \text{i.e.,} \quad \begin{vmatrix} 0 & \sin\alpha\,(\lambda+1) & \cos\alpha\,(1-\lambda) \\ 0 & \cos\alpha + \sin\alpha & \sin\alpha - \cos\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

i.e., $(\lambda+1)\sin\alpha\,(\sin\alpha - \cos\alpha) - (1-\lambda)\cos\alpha\,(\cos\alpha + \sin\alpha) = 0$

$$\lambda(\sin^2\alpha + \cos^2\alpha) + \sin^2\alpha - \cos^2\alpha - 2\sin\alpha\cos\alpha = 0$$

$$\lambda = \sin 2\alpha + \cos 2\alpha = \sqrt{2}\sin\left(2\alpha + \frac{\pi}{4}\right)$$

$$-1 \le \frac{\lambda}{\sqrt{2}} \le 1$$

$$-\sqrt{2} \le \lambda \le \sqrt{2}$$

for λ = 1,

$$\sin\left(2\alpha + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}\,; \quad \therefore \quad 2\alpha + \frac{\pi}{4} = \frac{\pi}{4}$$

General solution: $2\alpha + \frac{\pi}{4} = n\pi + (-1)^n\frac{\pi}{4}\,; \quad 2\alpha = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{4}$

If n is even, $2\alpha = n\pi$

odd, $2\alpha = n\pi - \frac{\pi}{2}$