LESSON 5

LIMITS, CONTINUITY AND DIFFERENTIABILITY

1. LIMIT OF A FUNCTION

The concept of limit is used to discuss the behaviour of a function close to a certain point.

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = L$$

Sometimes, functions approach different values as x-approaches x_0 from left and right. By left we mean $x < x_0$ and right means $x > x_0$. This is written as $x \to x_0^-$ and $x \to x_0^+$ respectively.

1. ALGEBRA OF LIMITS

(i) $\lim_{x \to a} (c_1 f(x) \pm c_2 g(x)) = \lim_{x \to a}(c_1 f(x)) \pm \lim_{x \to a}(c_2 g(x)) = c_1 l_1 \pm c_2 l_2$,

where c_1 and c_2 are given constants.

(ii) $\lim_{x \to a} f(x) . g(x) = \lim_{x \to a} f(x) . \lim_{x \to a} g(x) = l_1 . l_2$

(iii) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l_1}{l_2}, \quad l_2 \neq 0$

(iv) $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(l_2)$, if and only if $f(x)$ is continuous at $x = l_2$.

In particular, $\lim_{x\to a} \ln(g(x)) = \ln l_2$ if $l_2 > 0$.

(v) $\lim_{x\to a} (1+f(x))^{\frac{1}{g(x)}} = e^{\lim_{x\to a} \frac{\log(1+f(x))}{g(x)}}$

2. EVALUATION OF LIMITS

Following are indeterminate forms:

(i) $\frac{0}{0}$ (ii) $\frac{\infty}{\infty}$ (iii) $0 \times \infty$ (iv) $\infty - \infty$

(v) 0^0 (vi) ∞^0 (vii) 1^∞

Simplification:

(i) Direct substitution: We can directly substitute the number at which limit is to be find.

(ii) Rationalisation method

(iii) Factorization

(iv) Use of formulas

Example: **Find** $\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$.

Solution: This is apparently of the form ∞ minus ∞ and can be converted to $\frac{\infty}{\infty}$ form by multiplying numerator and the denominator by the conjugate.

$$\therefore \lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right) = \lim_{x\to\infty} \frac{(x-\sqrt{x^2+x})(x+\sqrt{x^2+x})}{x+\sqrt{x^2+x}}$$

$$= \lim_{x\to\infty} \frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}} = \lim_{x\to\infty}\left(\frac{-x}{x+\sqrt{x^2+x}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{-1}{1+\sqrt{1+\frac{1}{x}}}\right) = \frac{-1}{1+1} = \frac{-1}{2}$$

2. USE OF STANDARD LIMITS

These standard forms are used in case $f(x) \to 0$ when $x \to a$.

(i) $\lim_{x\to a} \frac{\sin f(x)}{f(x)} = 1$ (ii) $\lim_{x\to a} \cos f(x) = 1$

(iii) $\lim_{x\to a} \frac{\tan f(x)}{f(x)} = 1$ (iv) $\lim_{x\to a} \frac{\sin^{-1}(f(x))}{f(x)} = 1$

(v) $\lim_{x\to a} \frac{\tan^{-1}(f(x))}{f(x)} = 1$ (vi) $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$

Example: **Show that** $\lim_{x\to 0} \frac{\sin x(1-\cos x)}{x^3} = \frac{1}{2}$.

Solution: $\lim_{x\to 0} \frac{\sin x(1-\cos x)}{x^3} = \lim_{x\to 0} \frac{\sin x \cdot 2\sin^2 \frac{x}{2}}{x^3} = \lim_{x\to 0} 4\cos\frac{x}{2}\cdot\frac{\sin^3 \frac{x}{2}}{x^3} = \lim_{x\to 0} 4(1)\left(\frac{1}{2}\right)^3 = \frac{1}{2}$

SOME MORE STANDARD FORMS:

These standard forms are used in case $f(x) \to 0$ when $x \to a$.

(vi) $\lim_{x\to a} (1+f(x))^{\frac{1}{f(x)}} = e$ (vii) $\lim_{x\to a} \frac{b^{f(x)}-1}{f(x)} = \log_e b$ (b > 0)

(viii) $\lim_{x\to a} \frac{\log(1+f(x))}{f(x)} = 1$ (ix) $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$

Example: **Evaluate** $\lim_{x\to\infty} \left(\frac{x+6}{x+1}\right)^{x+4}$

Solution: The problem depends upon reducing the given expression to the form

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ which is *e*.

The given limit $= \lim_{x\to\infty}\left(\dfrac{1+\frac{6}{x}}{1+\frac{1}{x}}\right)^{x+4}$

$$= \frac{\lim_{x\to\infty}\left\{\left(1+\frac{6}{x}\right)^{\frac{x}{6}}\right\}^{\frac{(x+4)6}{x}}}{\lim_{x\to\infty}\left\{\left(1+\frac{1}{x}\right)^{x}\right\}^{\frac{x+4}{x}}} = \frac{e^6}{e^1},$$

(since $\dfrac{6(x+4)}{x} = 6 + \dfrac{24}{x}$ which tends to 6 and $\dfrac{x+4}{x} = 1 + \dfrac{4}{x}$ which tends to 1).

3. USE OF EXPANSION

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots..$

(ii) $a^x = 1 + \frac{x \log a}{1!} + \frac{x^2 (\log a)^2}{2!} + \ldots\ldots..$ $(a > 0)$

(iii) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots..$ $(-1 < x < 1)$

Example: **Evaluate** $\lim_{x \to 0} \frac{\log(5 + x) - \log(5 - x)}{x}$.

Solution: $\lim_{x \to 0} \frac{\log\left\{5\left(1 + \frac{x}{5}\right)\right\} - \log\left\{5\left(1 - \frac{x}{5}\right)\right\}}{x}$

$= \lim_{x \to 0} \frac{\log 5 + \log\left(1 + \frac{x}{5}\right) - \log 5 - \log\left(1 - \frac{x}{5}\right)}{x} = \lim_{x \to 0} \frac{\log\left(1 + \frac{x}{5}\right)}{5\left(\frac{x}{5}\right)} - \frac{\log\left(1 - \frac{x}{5}\right)}{-5\left(-\frac{x}{5}\right)}$

$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$

Example: **Evaluate** $\lim_{x \to 0} \frac{\tan^{-1} x - \sin^{-1} x}{\sin^3 x}$.

Solution: $\lim_{x \to 0} \frac{\left(x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right) - \left(x + \frac{1^2}{3!}\frac{x^3}{} + \frac{1^2\, 3^2}{5!} x^5 + \ldots\right)}{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\ldots..\right)^3}$

$$= \lim_{x\to 0} \frac{\left(-\left(\frac{1}{3}+\frac{1^2}{3!}\right)x^3+\left(\frac{1}{5}-\frac{1^2\,3^2}{5!}\right)x^5+......\right)}{x^3\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-.....\right)^3}$$

$$= \lim_{x\to 0} \frac{-\left(\frac{1}{3}+\frac{1^2}{3!}\right)+\left(\frac{1}{5}-\frac{1^2\,3^2}{5!}\right)x^2+......}{\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}.....\right)^3} = \frac{-1}{2}$$

4. CONTINUITY

1. CONTINUITY OF A FUNCTION

A function $f(x)$ is said to be continuous at $x = a$ if $\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$

If $f(x)$ is not continuous *a* $x = a$, we say that $f(x)$ is discontinuous at $x = a$.

- **Geometrical meaning of continuity**

Function $f(x)$ will be continuous at $x = c$ if there is no break in the graph of function $f(x)$ at the point $(c, f(c))$.

$f(x)$ will be discontinuous at $x = a$, in any of the following cases :

(i) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but are not equal.

(ii) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but not equal to $f(a)$.

(iii) $f(a)$ is not defined.

(iv) At least one of the limits does not exist.

Example: $f(x) = \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$ **for all x in** $\left(0, \frac{\pi}{2}\right)$ **except at** $x = \frac{\pi}{4}$. **Define** $f\left(\frac{\pi}{4}\right)$ **so that $f(x)$ may be continuous at** $x = \frac{\pi}{4}$.

Solution: $f(x)$ will be continuous at $x = \frac{\pi}{4}$ if $\lim_{x\to\pi/4} f(x) = f\left(\frac{\pi}{4}\right)$

$\therefore f\left(\frac{\pi}{4}\right)$ should be $= \lim_{x\to\pi/4} \frac{\sqrt{2}\cos x - 1}{\cot x - 1} = \lim_{x\to\pi/4} \frac{\left(\sqrt{2}\cos x - 1\right)\sin x}{\cos x - \sin x}$

$$= \lim_{x \to \pi/4} \frac{\left(\sqrt{2}\cos x - 1\right)\left(\sqrt{2}\cos x + 1\right)}{\left(\sqrt{2}\cos x + 1\right)\left(\cos x - \sin x\right)} \cdot \frac{\left(\cos x + \sin x\right)}{\left(\cos x + \sin x\right)} . \sin x$$

$$= \lim_{x \to \pi/4} \left(\frac{2\cos^2 x - 1}{\cos^2 x - \sin^2 x}\right) \frac{\left(\cos x + \sin x\right)\sin x}{\left(\sqrt{2}\cos x + 1\right)}$$

$$= \lim_{x \to \pi/4} \frac{\sin x\left(\cos x + \sin x\right)}{\sqrt{2}\cos x + 1}, \qquad \text{since } \cos^2 x + \sin^2 x = 1$$

$$= \frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)}{\sqrt{2} \cdot \frac{1}{\sqrt{2}} + 1} = \frac{1}{2}$$

- **Continuity in an open interval**

  A function *f*(*x*) is said to be continuous in an open interval (*a*, *b*) if it is continuous at each point of (*a*, *b*).

- **Continuity in a closed interval**

  A function *f*(*x*) is said to be continuous in an closed interval [*a*, *b*] if it is

  · continuous at each point (*a*, *b*)

  · *f*(*x*) is continuous from right at *x* = *a* i.e., $\lim_{x \to a^+} f(x) = f(a)$

  · *f*(*x*) is continuous from left at *x* = *b*

  i.e. $\lim_{x \to b^-} f(x) = f(b)$

- **Properties of continuous functions**

  Let *f* (*x*) and *g* (*x*) are continuous functions at *x* = *a*. Then

  (i) *c f* (*x*) is continuous at *x* = *a* where *c* is any constant

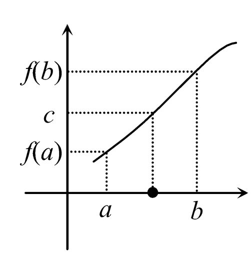
  (ii) *f* (*x*) ± *g* (*x*) is continuous at *x* = *a*

  (iii) *f* (*x*) . *g* (*x*) is continuous at *x* = *a*

  (iv) *f* (*x*)/*g* (*x*) is continuous at *x* = *a*, provided *g*(*a*) ≠ 0

- **Intermediate value theorem**

  If *c* is any real number between *f*(*a*) and *f*(*b*), then there exists at least one solution of the equation $f(x) = c$ in the open interval (*a, b*), if $y = f(x)$ is continuous in the interval.



- **Types of discontinuities**

Basically there are two types of discontinuity:

(i) Removable discontinuity

If $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$, then $f(x)$ has *a* removable discontinuity at $x = a$ and it can be removed by redefining $f(x)$ for $x = a$.

Example: **Redefine the function $f(x) = [x] + [-x]$ in such *a* way that it could become continuous for $x \in (0, 2)$.**

Solution: Here $\lim_{x\to 1} f(x) = -1$ but $f(1) = 0$.

Hence, $f(x)$ has *a* removable discontinuity at $x = 1$.

To remove this we define $f(x)$ as follows

$f(x) = [x] + [-x]$, $x \in (0, 1) \cup (1, 2)$ $= -1$, $x = 1$.

Now, $f(x)$ is continuous for $x \in (0, 2)$.

(ii) Non-removable discontinuity

If $\lim_{x\to a} f(x)$ does not exist, then we can not remove this discontinuity. So this become *a* non-removable discontinuity or essential discontinuity.

Example: **Prove that $f(x) = [x]$ has essential discontinuity at any $x \in I$.**

Solution: Proof is obvious as $\lim_{x\to a} f(x)$ does not exist for any $a \in I$.

Hence, $f(x) = [x]$ has essential discontinuity at any $x \in I$.

5. DIFFERENTIABILITY

The derivative of the function with respect to x is the function $f'(x)$ whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

provided the limit exists. i.e. $\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$

The function is said to be differentiable at $x = a$ if

Right hand derivative (RHD) at $x = a$ denoted by $f'(a + 0) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists and

Left hand derivative (LHD) at $x = a$ denoted by $f'(a - 0) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$ also exists.

In both these cases, we have assumed $h > 0$.

- **Differentiability in an interval**

(i) Differentiability in an open interval (a, b)

The function of $y = f(x)$ is said to be differentiable in (a, b) if it is differentiable at each point $x \in (a, b)$

(ii) In an closed interval $[a, b]$

The function $y = f(x)$, is said to be differentiable in $[a, b]$ if $f'(a + 0)$, $f'(b - 0)$ exist and $f'(x)$ exist for all $x \in (a, b)$.

Example: **If** $f(x) = \begin{cases} |x-1|([x]-x) & , \quad x \neq 1 \\ 0 & , \quad x = 1 \end{cases}$

Test the differentiability at $x = 1$, where [.] denotes the greatest integer function.

Solution: Check the differentiability at $x = 1$

$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \qquad (\because x > 1)$$

$$= \lim_{h\to 0} \frac{|1+h-1|([1+h]-(1+h)) - 0}{h}$$

$$= \lim_{h\to 0} \frac{h(1-1-h)}{h} = \lim_{h\to 0} \frac{h(-h)}{h} = 0$$

$$L(f'(1)) = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h} \qquad (x < 1)$$

$$= \lim_{h\to 0} \frac{f(1-h) - 0}{-h}$$

$$= \lim_{h\to 0} \frac{|1-h-1|([1-h]-(1-h)) - 0}{-h} = \lim_{h\to 0} \frac{h(0-1+h)}{-h} = 1$$

$Lf'(1) \neq Rf'(1)$

Hence $f(x)$ is not differentiable at $x = 1$.

- **Properties of differentiability**

Let $f(x)$ and $g(x)$ are differentiable functions at $x = a$. Then

(i) $c\, f(x)$ is differentiable at $x = a$ where c is any constant

(ii) $f(x) \pm g(x)$ is differentiable at $x = a$

(iii) $f(x) \,.\, g(x)$ is differentiable at $x = a$

(iv) $f(x)/g(x)$ is differentiable at $x = a$, provided $g(a) \neq 0$