LESSON 8

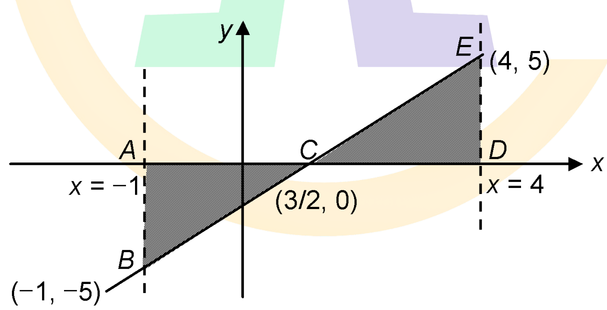
DEFINITE INTEGRATION

1. GEOMETRICAL INTERPRETATION OF DEFINITE INTEGRAL

Let $f(x)$ be a function defined on a closed interval $[a, b]$. Then $\int_a^b f(x)\,dx$ represents the algebraic sum of the areas of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$, $x = b$. Here algebraic sum means that area which is above the x-axis will be added in this sum with + sign and area which is below the x-axis will be added in this sum with − sign.

Example: **Evaluate:** $\int_{-1}^{4} (2x - 3)\,dx$.

Solution: $y = 2x - 3$ is a straight line, which lie below the x-axis in $\left[-1, \frac{3}{2}\right)$ and above in $\left(\frac{3}{2}, 4\right]$



Now area of $\Delta ABC = \frac{1}{2} \times \frac{5}{2} \times 5 = \frac{25}{4}$

Area of $\Delta CDE = \frac{1}{2} \times \frac{5}{2} \times 5 = \frac{25}{4}$

So $\int_{-1}^{4} (2x - 3)\,dx = -\frac{25}{4} + \frac{25}{4} = 0$

2. FUNDAMENTAL THEOREM OF CALCULUS

If $f(x)$ is a continuous function on $[a, b]$, then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ $(x \in [a, b])$

Example: **Evaluate:** $\int_0^1 \frac{dx}{\sqrt{2-x^2}}$.

Solution: $\int \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\frac{x}{\sqrt{2}} + c$

So $\int_0^1 \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\frac{x}{\sqrt{2}}\Big|_0^1 = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}(0)$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

3. GENERAL PROPERTIES OF DEFINITE INTEGRAL

1. $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt$

2. $\int_a^b f(x)dx \pm \int_a^b g(x)\,dx = \int_a^b \left(f(x) \pm g(x)\right)dx$

3. $\int_a^b f(x)dx = \int_a^b f(t)\,dt = \int_a^b f(y)dy$

4. $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$

Example: **Evaluate** $\int_2^3 \frac{dx}{x\sqrt{4x^2+1}}$.

Solution: $I = \int_2^3 \frac{dx}{x\sqrt{4x^2+1}}$

Put $x = \frac{1}{t} \Rightarrow dx = -\frac{dt}{t^2}$

So $I = \int_{1/2}^{1/3} \frac{-dt}{t^2\left(\frac{1}{t}\right)\sqrt{\frac{4}{t^2}+1}} = -\int_{1/2}^{1/3} \frac{dt}{\sqrt{4+t^2}}$

$= \int_{1/3}^{1/2} \frac{dt}{\sqrt{4+t^2}} = \ln\left(t+\sqrt{4+t^2}\right)\Big|_{1/3}^{1/2} = \ln\left(\frac{3}{2}\left(\frac{\sqrt{17}+1}{\sqrt{37}+1}\right)\right)$

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5. $\displaystyle\int_a^b f(x)dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)dx + ... + \int_{c_n}^b f(x)dx$

Example: **Evaluate** $\displaystyle\int_{-2}^{3} \left|x^2-1\right|\,dx$.

Solution: $\displaystyle\int_{-2}^{3}\left|x^2-1\right|\,dx = \int_{-2}^{-1}\left|x^2-1\right|\,dx + \int_{-1}^{1}\left|x^2-1\right|\,dx + \int_{1}^{3}\left|x^2-1\right|\,dx$

(Here modulus function will change at the points, when $x^2 - 1 = 0$ i.e. at $x = \pm 1$)

So $I = \displaystyle\int_{-2}^{-1}(x^2-1)\,dx + \int_{-1}^{1}(1-x^2)\,dx + \int_{1}^{3}(x^2-1)\,dx$

$= \left.\frac{x^3}{3}-x\right|_{-2}^{-1} + \left.x-\frac{x^3}{3}\right|_{-1}^{1} + \left.\frac{x^3}{3}-x\right|_{1}^{3}$

$= \left(\frac{-1}{3}+1\right) - \left[\frac{-8}{3}+2\right] + \left[1-\frac{1}{3}\right] - \left[-1+\frac{1}{3}\right] + \left[\frac{27}{3}-3\right] - \left[\frac{1}{3}-1\right]$

$= \frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+6+\frac{2}{3} = \frac{28}{3}$

6. $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Example: **Evaluate** $\displaystyle\int_2^7 \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{9-x}}$.

Solution: $$I = \int_2^7 \frac{\sqrt{x}\ dx}{\sqrt{x} + \sqrt{9-x}} \qquad \ldots(i)$$

$$I = \int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x} + \sqrt{9-(9-x)}}\, dx$$

$$I = \int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x} + \sqrt{x}}\, dx \qquad \ldots(ii)$$

Adding (i) and (ii), we get

$$2I = \int_2^7 \left(\frac{\sqrt{x}}{\sqrt{x} + \sqrt{9-x}} + \frac{\sqrt{9-x}}{\sqrt{x} + \sqrt{9-x}} \right) dx = \int_2^7 dx = x \Big|_2^7 = 7 - 2 = 5$$

So $I = \dfrac{5}{2}$

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7. $$\int_{-a}^{a} f(x)\, dx = \int_0^a (f(x) + f(-x))\, dx$$

Example: **Evaluate** $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$.

Solution: $$I = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$$

Here $f(x) = \dfrac{1}{(1+e^x)(1+x^2)}$

$\Rightarrow$ $f(-x) = \dfrac{1}{(1+e^{-x})(1+(-x)^2)} = \dfrac{e^x}{(1+e^x)(1+x^2)}$

$$f(x) + f(-x) = \frac{1}{1+x^2}\left[\frac{1}{1+e^x} + \frac{e^x}{1+e^x}\right] = \frac{1}{1+x^2}$$

So $I = \displaystyle\int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^{\sqrt{3}} = \frac{\pi}{3}$

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8. $$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$$

9. $$\int_a^{b} f(x)\,dx = (b-a)\int_0^{1} f((b-a)x+a)\,dx$$

Example: **Evaluate** $\displaystyle\int_0^{\pi}\frac{x\,dx}{1+\cos^2 x}$.

Solution:

$$I=\int_0^{\pi}\frac{x\,dx}{1+\cos^2 x}$$

$$I=\int_0^{\pi}\frac{(\pi-x)\,dx}{1+\cos^2(\pi-x)}=\int_0^{\pi}\frac{(\pi-x)\,dx}{1+\cos^2 x}$$

Adding both, we get

$$2I=\int_0^{\pi}\frac{\pi\,dx}{1+\cos^2 x}$$

$\Rightarrow$ $$I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{1+\cos^2 x}$$

$$=\frac{\pi}{2}\left[\int_0^{\pi/2}\frac{dx}{1+\cos^2 x}+\int_0^{\pi/2}\frac{dx}{1+\cos^2(\pi-x)}\right]$$

$$=\frac{\pi}{2}\left[2\int_0^{\pi/2}\frac{dx}{1+\cos^2 x}\right]=\pi\int_0^{\pi/2}\frac{dx}{1+\cos^2 x}$$

$$=\pi\int_0^{\pi/2}\frac{\sec^2 x\,dx}{2+\tan^2 x}$$ put $\tan x = t$

$$I=\pi\int_0^{\infty}\frac{dt}{t^2+2}=\frac{\pi}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\Bigg|_0^{\infty}=\frac{\pi^2}{2\sqrt{2}}$$

4. PERIODIC PROPERTIES OF DEFINITE INTEGRAL

1. If $f(x)$ is *a* periodic function with period *p*, then $\int_a^{a+np} f(x)\,dx = n\int_0^p f(x)dx, n \in I$

2. If $f(x)$ is *a* periodic function with period *p*, then

$$\int_{mp}^{np} f(x)\,dx = (n-m)\int_0^p f(x)dx\ ,\quad n, m \in I$$

3. If $f(x)$ is *a* periodic function with period *p*, then $\int_{a+np}^{b+np} f(x)\,dx = \int_a^b f(x)dx\ ,\quad n \in I$

Example: **Evaluate** $\int_{10\pi+\frac{\pi}{6}}^{10\pi+\frac{\pi}{3}} (\sin x + \cos x)\,dx$.

Solution: $f(x) = \sin x + \cos x$ is periodic with period 2π.

Let $I = \int_{10\pi+\frac{\pi}{6}}^{10\pi+\frac{\pi}{3}} (\sin x + \cos x)\,dx = \int_{\pi/6}^{\pi/3} (\sin x + \cos x)\,dx$

$$= (\sin x - \cos x)\Big|_{\pi/6}^{\pi/3}$$

$$= \left[\frac{\sqrt{3}}{2} - \frac{1}{2}\right] - \left[\frac{1}{2} - \frac{\sqrt{3}}{2}\right] = (\sqrt{3} - 1)$$

8. DEFINITE INTEGRAL AS THE LIMIT OF A SUM

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + \left(\frac{b-a}{n}\right)r\right)$$

Example: **Evaluate** $\lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}} + \frac{1}{\sqrt{4n^2-4}} + \frac{1}{\sqrt{4n^2-9}} + \ldots\ldots \frac{1}{\sqrt{3n^2}}\right]$.

Solution:

$$L = \lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\frac{1}{\sqrt{4n^2-9}}+.......\frac{1}{\sqrt{3n^2}}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\frac{1}{\sqrt{4-(r/n)^2}}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{(1-0)}{n}\frac{1}{\sqrt{4-\left(0+r.\left(\frac{1-0}{n}\right)\right)^2}}$$

, which is of the form

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{b-a}{n}f\left(a+r\left(\frac{b-a}{n}\right)\right)$$

. Here b = 1, a = 0 and $f(x)=\frac{1}{\sqrt{4-x^2}}$

So $L=\int_0^1\frac{dx}{\sqrt{4-x^2}}=\sin^{-1}\frac{x}{2}\Big|_0^1=\frac{\pi}{6}$