LESSON 9

AREAS BOUNDED BY CURVES

1. APPLICATION OF INTEGRATION TO AREAS

Definite integral is used to evaluate areas bounded by curves.

Guidelines

(i) Check whether the curve is symmetrical about the x-axis or not. The curve is symmetrical about the x-axis, if its equation is unchanged when y is replaced by $-y$.

(ii) The curve is symmetrical about the y-axis if its equation is unchanged when x is replaced by $-x$.

(iii) Put $y = 0$ in the equation of the curve. This will give the points where it cuts the x-axis

(iv) Put $x = 0$ in the equation of the curve. This will give the points where it cuts the y-axis.

(v) The curve is symmetrical about the line $y = x$ if its equation does not change when x and y are interchanged.

(vi) Find the turning points of the graph by equating $\frac{dy}{dx} = 0$

(vii) Find the intervals of curve in which it increases and decreases if required.

(viii) Use periodicity wherever possible.

(ix) Check behaviour at $x \to \pm\infty$ and $y \to \pm\infty$.

Example: **Trace the curve $y^2(2a - x) = x^3$, $a > 0$.**

***Solution*:** Note that the curve passes through the origin and is symmetrical about the x-axis.

$$y^2 = \frac{x^3}{2a - x}$$

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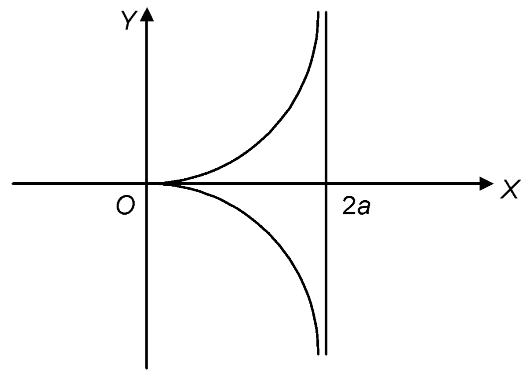
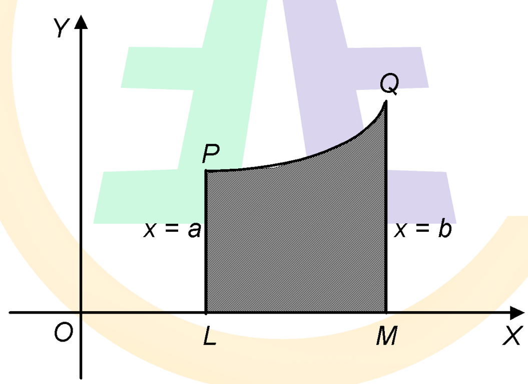


L.H.S. is positive. If x is negative or if x is greater than $2a$, R.H.S. becomes negative. Hence the curve lies only in the interval 0 to $2a$. When $x \to 2a$, $y \to \infty$. Therefore the line $x = 2a$ is an asymptote for the curve. A rough Figure is shown.

2. ESTIMATION OF AREAS

Four cases are discussed below:

Case I : PQ is an arc of a curve whose equation is $y = f(x)$. We have an area bounded by PQ on one side; by the x-axis on another and the two parallel lines $x = a$ and $x = b$ (shown by PL and QM), $a < b$.



The area $PLMQ = \int_{x=a}^{x=b} y\,dx = \int_a^b f(x)\,dx$

Case II: PQ is an arc of a curve whose equation is $y = f(x)$ or $x = f(y)$.

In this case y-axis is one boundary and the other two are the lines $y = c$ and $y = d$.

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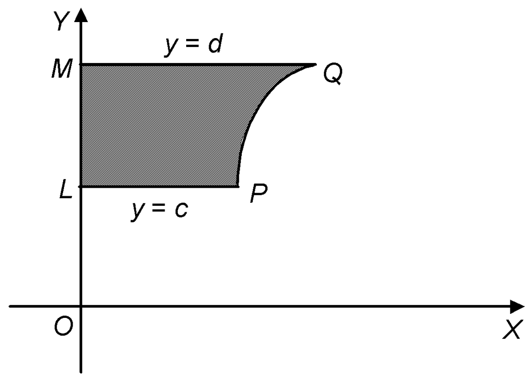
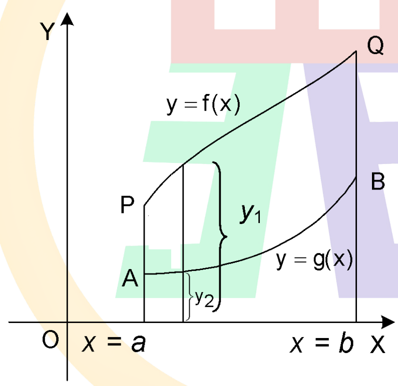


The area $LPQM = \int_{y=c}^{y=d} x\,dy = \int_c^d f(y)\,dy$

In this case the integration is with respect to *y*.

Case III: The figure encloses an area between two curves one of which is represented by *PQ* with equation *y* = *f*(*x*) and the other by *AB* with the equation *y* = *g*(*x*).



Area $PABQ = \int_a^b (y_1 - y_2)\,dx$ where $y_1 = f(x)$ and $y_2 = g(x)$

$$= \int_a^b \{f(x) - g(x)\}\,dx$$

Case IV: The figure represents the region bounded by a closed curve ACQBP.

The area of the region bounded by a closed curve ACQBP is $\int_a^b (y_1 - y_2)\,dx\,,\ y_1 > y_2$

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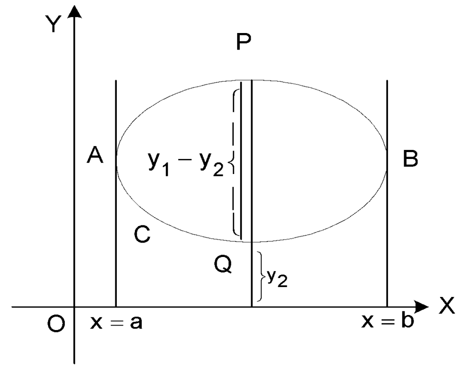


Fig. 4

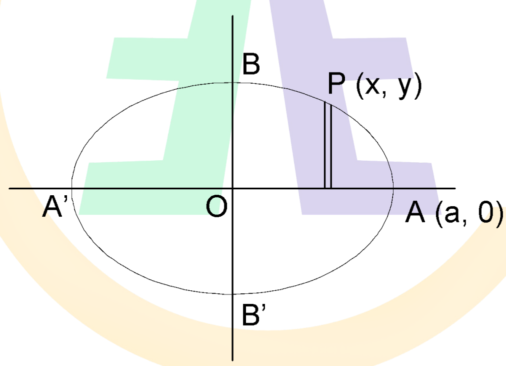
The values of y_1 and y_2 are obtained by solving the equation of the curve as a quadratic in y whose larger root y_1 and smaller root y_2 are functions of x.

a and b are the coordinates of the points of contact of tangents drawn parallel to the y-axis.

Example: **Find the area of the ellipse** $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0}$

***Solution*:** The ellipse is symmetrical about both axes and hence the area enclosed

= 4 (area of the quadrant)



$$= 4\int_0^a y\,dx = 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = \frac{4b}{a}\int_0^a \sqrt{\left(a^2 - x^2\right)}dx = \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[\frac{a^2\pi}{4}\right] = \pi ab \text{ sq. units}$$

Note: Sometimes it is better to use the formula $\int_c^d x\,dy$ instead of $\int_a^b y\,dx$ in the computation of area to simplify calculations, as the following illustration shows.