LESSON 10

DIFFERENTIAL EQUATIONS

1. INTRODUCTION

Generally, any equation, such as

$f(x, y, a) = 0$ .... (i)

represents for each individual value of a, a member of a family of curves.

From the given equation, solve for a, and the equation $\phi(x, y) = a$ may be obtained; and on differentiating, 'a' gets removed. The resulting equation involving $\frac{dy}{dx}$ is known as a differential equation i.e. the equation representing all the members of the family $f(x, y, a) = 0$ or alternately $\phi(x, y) = a$.

2. DIFFERENTIAL EQUATION

An equation involving an independent variable x, a dependent variable y and the differential coefficients of the dependent variable i.e. $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, ……etc is known as a differential equation. It can also be expressed as a function of variables x, y and derivatives of y w.r.t. x such as

$$f\left(x, y, \frac{dy}{dx}\right) = 0$$

Geometrically, differential equations represent a family of curves having a common property.

3. FORMATION OF DIFFERENTIAL EQUATION

To form a differential equation, we differentiate the given family of curves and eliminate the unknown constants as follows:

(i) Consider the equation $y = ax$. This represents the Cartesian equation to a family of straight lines through the origin.

Differentiating $y = ax$, we get $\frac{dy}{dx} = a.$ Eliminating a, we get the differential equation $y = \frac{dy}{dx} \cdot x$.

Hence $y = x\frac{dy}{dx}$ is the differential equation of all straight lines passing through the origin.

(ii) Consider another example, the equation $x^2 + y^2 = a^2$. This, for various a, represents a family of concentric circles with centre at origin.

Differentiating the relation we get

$$2x + 2y\frac{dy}{dx} = 0$$ (*a* is eliminated)

i.e. $x + y\frac{dy}{dx} = 0$

which may be said to be the differential equation to a family of concentric circles.

(iii) Now consider another equation representing a family of curves in the form

$$f(x, y, a, b) = 0 \quad \text{.... (i)}$$

containing two arbitrary constants. In this case, since there are two constants, it becomes necessary to differentiate equation twice so that the result contains $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and can be expressed in the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \quad \text{.... (ii)}$$

This equation is said to represent a differential equation of the family of curves represented by equation (i). Thus in the case of $y = ax + b$

$$\frac{dy}{dx} = a$$

$\frac{d^2y}{dx^2} = 0$ which is the differential equation of the family of all straight lines.

Differential equations

Example: **Form the differential equation of the relation $x^2 + y^2 = 2ax$.**

Solution: Consider the relation $x^2 + y^2 = 2ax$

Differentiating, $2x + 2y\frac{dy}{dx} = 2a$

Eliminating a, $x^2 + y^2 = x\left(2x + 2y\frac{dy}{dx}\right) \Rightarrow x^2 - y^2 + 2xy\frac{dy}{dx} = 0$

In this case the relation contains only one constant and hence the differential equation contains only $\frac{dy}{dx}$.

4. ORDER AND DEGREE OF DIFFERENTIAL EQUATION

1. ORDER OF A DIFFERENTIAL EQUATION

The highest derivative occurring in a differential equation defines its order.

2. DEGREE OF A DIFFERENTIAL EQUATION

The power of the highest order derivative occurring in a differential equation is called the degree of the differential equation, for this purpose the differential equation is made free from radicals and fractions of derivatives.

EXAMPLES

Differential equation	Order of D.E.	Degree of D.E.
$\frac{dy}{dx} + 4y = \sin x$	1	1
$\left(\frac{d^2y}{dx^2}\right)^4 + \left(\frac{dy}{dx}\right)^5 - y = e^x$	2	4
$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3y = \cos x$	2	1
$\frac{dy}{dx} = \frac{x^4 - y^4}{xy\,(x^2 + y^2)}$	1	1
$y = x\frac{dy}{dx} + \sqrt{a^2\left(\frac{dy}{dx}\right)^2 + b^2}$		

$\Leftrightarrow$ $(x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + (y^2 - b^2) = 0$ 1 2

. $\frac{d^2y}{dx^2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}$

$\Leftrightarrow$ $\left(\frac{d^2y}{dx^2}\right)^2 - \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3 = 0$ 2 2

5. SOLUTION OF A DIFFERENTIAL EQUATION

1. EQUATIONS WITH SEPARABLE VARIABLE

Differential equations of the form $\frac{dy}{dx} = f(x, y)$

can be reduced to form $\frac{dy}{dx} = g(x)\,h(y)$

where it is possible to take all terms involving *x* and *dx* on one side and all terms involving *y* and *dy* to the other side, thus separating the variables and integrating.

Example: **Solve the differential equation** $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$.

Solution: Separating the variables $\frac{dy}{dx} = e^{-y}\,(e^x + x^2)$

$e^y dy = (e^x + x^2)\,dx$, integrating, the solution is

$e^y = e^x + \frac{x^3}{3} + A$

$\Rightarrow$ $3\,(e^y - e^x) = x^3 + C$ (*C* is an arbitrary constant)

2. EQUATIONS REDUCIBLE TO EQUATIONS WITH SEPARABLE VARIABLE

A differential equations of the form $\frac{dy}{dx} = f(ax + by + c)$

can not be solved by separating the variables directly. By substituting *ax* + *by* + *c* = *t* and $a + b\frac{dy}{dx} = \frac{dt}{dx}$, the differential equation can be separated in terms of variables *x* and *t*.

Differential equations

Example: **Solve the differential equation** $\frac{dy}{dx} = \cos(x+y)$.

Solution: Put $x + y = t$

$\Rightarrow \quad 1 + \frac{dy}{dx} = \frac{dt}{dx}$

Thus $\frac{dt}{dx} - 1 = \cos t \quad \Rightarrow \quad \frac{dt}{1+\cos t} = dx$

Integrating both sides

$\Rightarrow \quad \tan\left(\frac{t}{2}\right) = x + c \quad \Rightarrow \quad \tan\left(\frac{x+y}{2}\right) = x + c$, where c is the parameter.

3. HOMOGENEOUS DIFFERENTIAL EQUATIONS

A differential equations of the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$

where f_1 (x, y) and f_2 (x, y) are homogeneous functions of x and y of the same degree, is called a homogeneous equation.

It can also be written in form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, by dividing both the functions by x^n where n is the degree of function.

To solve this equation, substitute

$\frac{y}{x} = t$ or $y = tx$

$\Rightarrow \quad \frac{dy}{dx} = t + x\frac{dt}{dx}$

Then the equation reduces to $t + x\frac{dt}{dx} = f(t)$ which can be easily reduced to variable separable as $\frac{dt}{f(t) - t} = \frac{dx}{x}$.

Example: **Solve the differential equation** $\frac{dy}{dx} = \frac{x-y}{x+y}$.

(Note: $x - y$, $x + y$ are homogeneous in x and y of degree one)

Solution: Taking $y = vx$,

$$\frac{dy}{dx} = v \cdot 1 + x\frac{dv}{dx}$$

Substituting in the given equation

$$v + x\frac{dv}{dx} = \frac{x - vx}{x + vx} = \frac{1 - v}{1 + v}$$

$$x\frac{dv}{dx} = \frac{1 - v}{1 + v} - v = \frac{1 - v - v - v^2}{1 + v}$$

Now, separating the variables

$$\int \frac{1 + v}{1 - 2v - v^2}\, dv = \int \frac{dx}{x} + A$$

$$-\frac{1}{2}\log(1 - 2v - v^2) = \log x + A$$

$\log[(1 - 2v - v^2)x^2] = \text{constant}$; $(1 - 2v - v^2)x^2 = C$.

$x^2 - 2xy - y^2 = C$ is therefore the solution where C is an arbitrary constant.

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4. EQUATIONS REDUCIBLE TO HOMOGENEOUS EQUATION

A differential equation of the form $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$

where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, can be reduced to homogeneous equation by putting $x = X + h$ and $y = Y + k$. where h, k are such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$ also

$$\frac{dy}{dx} = \frac{dY}{dX}$$

hence equation reduces to $\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$ (homogeneous form).

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda$, then $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} = \frac{\lambda(a_1x + b_1y) + c_1}{(a_1x + b_1y) + c_2}$ can be solved by putting $a_1x + b_1y = t$, as then it reduces to equation with variable separable.

Example: **Solve the differential equation** $\mathbf{\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}}$.

Solution: Take $x = X + l$; $y = Y + m$ (l and m are constants) $\frac{dy}{dx} = \frac{dY}{dX}$

$\therefore$ the equation becomes (in X, Y)

$$\frac{dY}{dX} = \frac{X + 2Y + \ell + 2m - 3}{2X + Y + 2\ell + m - 3} = \frac{X + 2Y}{2X + Y}$$ if ℓ, m are chosen to satisfy

$$\left.\begin{array}{l} \ell + 2m - 3 = 0 \\ 2\ell + m - 3 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \ell = 1 \\ m = 1 \end{array}\right\}$$

In X, Y the equation is homogeneous and of the first degree. Set $Y = VX$

$$V + X\frac{dV}{dX} = \frac{X + 2VX}{2X + VX} = \frac{1 + 2V}{2 + V}$$

$$X\frac{dV}{dX} = \frac{1 + 2V - (2 + V)V}{2 + V} = \frac{1 - V^2}{2 + V}$$

Separating the variables (X, V) and integrating,

$$\int \frac{2 + V}{1 - V^2}\, dV = \int \frac{dX}{X} + A$$, where A is an arbitrary constant.

$$\int \left(\frac{1}{2}\cdot\frac{1}{1 + V} + \frac{3}{2}\cdot\frac{1}{1 - V}\right) dV = \int \frac{dX}{X} + A$$

$$\frac{1}{2}\log(1 + V) - \frac{3}{2}\log(1 - V) - \log X = A$$

Now $V = \dfrac{Y}{X} = \dfrac{y - m}{x - \ell} = \dfrac{y - 1}{x - 1}$

Reverting to x and y, the solution is

$$\frac{1}{2}\log\left(1 + \frac{y - 1}{x - 1}\right) - \frac{3}{2}\log\left(1 - \frac{y - 1}{x - 1}\right) - \log(x - 1) = A$$

which simplifies to $\left[\dfrac{x + y - 2}{(x - 1)(x - 1)^2}\cdot\dfrac{(x - 1)^3}{(x - y)^3}\right] = C\left(= e^{2A}\right)$

$(x + y - 2) = C\,(x - y)^3$ (C is an arbitrary constant)

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5. LINEAR DIFFERENTIAL EQUATIONS

A differential equation of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

where $P(x)$ and $Q(x)$ are functions of x only or constants, is known as linear differential equation.

The solution of differential equation is given by

$$y\,(I.F.) = \int Q(x)\,(I.F)\,dx$$

Example: **Solve the differential equation** $\boldsymbol{x\dfrac{dy}{dx} = y - \cos\left(\dfrac{1}{x}\right)}$.

Solution: Here, $x\dfrac{dy}{dx} - y = -\cos\left(\dfrac{1}{x}\right)$

$\therefore$ $\dfrac{dy}{dx} - \dfrac{1}{x}\,.\,y = -\dfrac{1}{x}\cos\left(\dfrac{1}{x}\right)$; this is in the linear form.

Integrating factor $e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1}$.

Multiplying by the integrating factor,

$$\frac{1}{x}\cdot\frac{dy}{dx} - \frac{1}{x^2}y = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right) \quad \text{or} \quad \frac{d}{dx}\left\{\frac{1}{x}\,.\,y\right\} = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$$

or $d\left(\dfrac{y}{x}\right) = -\dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right)dx$

or $\displaystyle\int d\left(\frac{y}{x}\right) = \int -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$

$\therefore$ $\displaystyle\frac{y}{x} = \int \cos\left(\frac{1}{x}\right)d\left(\frac{1}{x}\right)$

or $\dfrac{y}{x} = \sin\left(\dfrac{1}{x}\right) + c$, where c is an arbitrary constant.

$\therefore$ $y - x\sin\left(\dfrac{1}{x}\right) = cx$.

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6. EQUATIONS REDUCIBLE TO THE LINEAR DIFFERENTIAL EQUATION

(i) If equation is of the form.

$R(y)\dfrac{dy}{dx} + P(x)\,S(y) = Q(x)$ such that $\dfrac{dS}{dy} = R$, then put $S(y) = t$

$\Rightarrow$ $\dfrac{dt}{dx} = \dfrac{dS}{dx} = \dfrac{dS}{dy}\cdot\dfrac{dy}{dx} = \dfrac{Rdy}{dx}$

Thus differential equation reduces to $\dfrac{dt}{dx} + P(x)t = Q(x)$

which is linear differential equation.

Example: **Solve the differential equation** $\cos 2y \dfrac{dy}{dx} + \dfrac{1}{x}\sin 2y = e^x$.

Solution: Put sin2*y* = *t*

$\Rightarrow \quad 2\cos 2y \dfrac{dy}{dx} = \dfrac{dt}{dx} \quad \Rightarrow \quad \cos 2y \dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{dt}{dx}$

so given equation reduces to $\dfrac{1}{2}\dfrac{dt}{dx} + \dfrac{1}{x}t = e^x \quad \Rightarrow \quad \dfrac{dt}{dx} + \dfrac{2}{x}t = 2e^x$

I.*F*. = $e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$

$\therefore$ solution is $tx^2 = 2\int x^2 e^x dx$

$\Rightarrow$ (sin*y*)x^2 = 2(x^2e^x − 2xe^x + 2e^x) + *c*, where '*c*' is an arbitrary constant.